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FINANCING AND INITIAL OPERATIONS OF NEW FIRMS

GEORGE W. SUMMERS

Associate Professor, General Business
Arizona State University

1962

PRENTICE-HALL, INC.
Englewood Cliffs, N. J.

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1853

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L. C. Catalog Card Number: 62:13507

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Foreword

This volume is one of five doctoral dissertations selected for publication in the third annual Doctoral Dissertation Competition sponsored by the Program in Economic Development and Administration of The Ford Foundation. The winning dissertations were completed during the academic year 1960–61 by doctoral candidates in business administration, the social sciences and other fields relevant to the study of problems of business.

The dissertation competition is intended to recognize and encourage excellence in research on business by graduate students. In specific pursuit of this aim, publication awards are made to persons recently granted doctorates in business whose dissertation research is especially distinguished by its analytical content and strong roots in underlying disciplines. To the same end, awards are made to a selected number of persons outside business schools who in their doctoral dissertations pursued with distinction interests relevant to the field of business.

The dissertations selected include, in addition to Dr. Summers', the following:

Portfolio Selection: A Simulation of Trust Investment
Geoffrey P. E. Clarkson
Graduate School of Industrial Administration
Carnegie Institute of Technology

The Investment Decision Under Uncertainty: Portfolio Selection
Donald E. Farrar
Faculty of Arts and Sciences
Harvard University

An Evaluation of a Forced-Choice Differential Accuracy Approach to the Measurement of Supervisory Empathy
Richard S. Hatch
Department of Psychology
University of Minnesota

The Term Structure of Interest Rates
David Meiselman
Department of Economics
University of Chicago

On behalf of The Ford Foundation, I wish to express my gratitude to the members of the Editorial Committee for the care and thought they devoted to the selection process. The three members of this Committee, who made the final selection of winning dissertations, are: Professor Alex Bavelas of Stanford University, Professor Roy Blough of Columbia University, and Professor Robert Ferber of the University of Illinois.

The Editorial Committee's task was considerably lightened by the assistance of eleven readers, experts in the wide range of disciplines covered in the competition, who carefully screened the dissertations submitted. The Foundation joins the Committee in acknowledging their debt to Professors Paul Breer and Henry A. Landsberger of Cornell University, Richard M. Cyert of the Carnegie Institute of Technology, Robert B. Fetter of Yale University, Samuel Goldberg of Oberlin College, Albert H. Hastorf of Stanford University, Austin C. Hoggatt and Lyman W. Porter of the University of California (Berkeley), Daniel M. Holland of the Massachusetts Institute of Technology, John Neter of the University of Minnesota, and Harry V. Roberts of the University of Chicago.

Finally, my colleagues and I wish to acknowledge the important contribution of Prentice-Hall, Inc., to the publication of the selected dissertations.

DYKE BROWN
VICE PRESIDENT
THE FORD FOUNDATION

New York, New York
January, 1962

Acknowledgments

I am especially grateful to Dr. John D. C. Little on two counts. In course work and on industrial research projects he did much to illustrate the nature of scientific modeling in decision processes and thereby to give the field of operations research greater meaning. In addition, he served as advisor for this study. His patient and thorough review and his many constructive suggestions both as to organization and content are deeply appreciated.

To Dr. Russell L. Ackoff and Dr. E. Leonard Arnoff for providing the environment and leadership whereby class room and project experience in operations research can be gained, I express my gratitude. Their conviction as to the importance of the field did much to stimulate my initial interest in the problems considered in this study.

George W. Summers

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FINANCING AND
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CHAPTER 1

Introduction

No matter what ambitions a person may wish to satisfy by founding a business, the central prerequisite with which he must concern himself is cash flow and its static counterpart — cash position. In a firm, cash or equivalent is converted into productive equipment, materials, and services which are, in their turn, converted into products or other services and put on the market. In the market products are, with some time lag, re-converted to cash which returns to the firm for a repetition of the cycle. Typically, all phases of the cycle are in process within any one short period of time. Consequently, a number of virtually simultaneous demands are being made for payment from the firm and, during the same period, funds are flowing into the firm's possession from a variety of sources. The cash equivalent of inputs by all creditors through time must be followed by at least an equivalent counterflow of cash from markets. Instantaneous balance is, of course, not required but the net cash accumulated in previous periods and during the current period must be at least as large as the demands for cash during the current period. If not, liquidity, the short-term ability of the firm to meet its debts, is impaired. If the prospects for the future net flow of cash are favorable, this short-term impairment can be overcome with the payment of a penalty. If not, solvency, the long-term ability of the firm to meet its debts, is impaired and bankruptcy follows.

After assembly and organization of the necessary productive resources, every new firm commences sales operations with what should be the worst cash flow circumstance of its existence: heavy expense and little or no income. This study concerns itself with this particularly critical period in the life of the firm. Its purpose is to investigate the structure and relationships between the major forces which affect cash position through time during this initial period.

1.1. Organization of a New Firm

An exhaustive list of the tasks which must be accomplished and the decisions which must be made in establishing a new firm is inappropriate. Nonetheless, pointing out the major tasks will serve to put the body of this study in perspective. First, the means by which the new firm intends to garner revenue must be decided. The results of product development and service design are then subjected to marketing and sales promotion review concurrent with an effort to decide the location of markets to be served, of plant, and of channels of distribution. The sources and organization of physical production resources must be planned. Financing requirements, legal considerations, and management principles provide the basis for the structural and functional organization plans of the firm. Feasibility having been demonstrated, the plans are converted to a time schedule to integrate the actual formative activities. When set-up has been completed, plant operations are begun; either accompanied or followed by further product testing, market research, and sales promotion activity. Through the entire set-up period the planning, research, testing, and pulling together of men and resources have occasioned investment, borrowing, and expense no part of which has been offset by revenue.

This study concentrates on the flows and processes of production, physical demand, expense, and revenue during the period which begins at the commencement of sales operations and, if the firm succeeds, ceases either when relatively stable levels of production, demand, and positive net revenue flow have been established or when loans have been repaid, whichever is later. The emphasis is upon manufacturing firms characterized by production technologies which fulfill the requirements of segmentation. The latter is a term credited to Joel Dean[1] and refers to production processes in which changes in output rate over a wide range are accomplished by adding or subtracting identical man-machine units to or from the work force. The position is taken that, having made choices as to product, location, methods of distribution, and form of organization, sources such as engineering estimates and real estate appraisals can provide reasonably accurate forecasts of set-up expense. It is maintained, however, that an adequate method for determining the amount of capital needed to offset early operating losses and for determining the best initial production schedule to follow has not been available.

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[1] Joel Dean, *Managerial Economics*, Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1951, 275.

1.2. Importance of the Problem

A study undertaken by the research staffs of the Federal Reserve System represents the most complete compilation of evidence on the points at issue which has been obtained to date.[2] While it is true that the report concentrates on small business, firms with less than 100 employees account for some 99 per cent of all firms in the economy.[3] In fact, any of the common measures of size demonstrates that our business population is comprised of a great many small firms which together account for less than half of the gross national product.[4] However, in a discussion concerned with the problem of getting started, the small business sub-group is not only relevant but virtually the entire population.

In the report, data which have been compiled from several sources are presented on business failures.[5] They show that the net increase in business population, 1955 to 1956, was 42,000 firms. However, 380,000 businesses had to be formed that year to offset the 338,000 discontinuances, exclusive of transfers, and to provide the 42,000 firm excess. While not all discontinuances can be considered failures, the Department of Commerce is quoted as stating that business failures involving loss to creditors occurred in 1940 at the rate of 97 per 1,000 firms in operation at the beginning of the year and 79 per 1,000 in 1956.[6] In the latter year an average of 4.3 million firms were in operation so nearly 340,000 firms failed. Of these, approximately 60 per cent were five or less years old, about 90 per cent had liabilities of $100,000 or less; 27 per cent had liabilities of between $25,000 and $100,000.

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[2] *Financing Small Business,* Report to the Committees on Banking and Currency and the Select Committees on Small Business, 85th U. S. Congress, 2nd Session, by the Federal Reserve System, Parts I and II, 1958.

[3] *Ibid.,* 2.

[4] *Ibid.,* 170, 171.

[5] *Ibid.,* 211–214.

[6] *Ibid.,* 211.

One interpretation of the evidence on failures above could be that, in an economy in which anyone is free to establish a business provided that he fulfills some modest legal requirements, it is not surprising to find that so many fail. One could expect a portion of would-be businessmen to be possessed of more courage than acumen and a second portion to be the victims of highly unusual deviations from expectations. It may be that the foolhardy go forward in spite of the advice of those whose business it is to judge the soundness of proposed ventures and to back their judgment by lending funds entrusted to their care. Nevertheless, when nearly 10 per cent of the business population fail every year, it would appear that planning methods can stand improvement.

If it is the case that bankers generally know how to evaluate business ventures with less error than the owners, then one would expect low default rates on bank loans to business. The report does not present direct evidence on per cent of loans in default for commercial banks. However, these data are presented for Reconstruction Finance Corporation loans as of December 31, 1951.[7] They show that 13.2 per cent of a sample of all direct business loans were in default on that date. While it is true that RFC loans were confined predominantly to businesses unable to get credit from the usual private sources on reasonable terms, the RFC did insist on adequate collateral and was required by law to use sound banking practices in evaluating applications. Typically, those requesting loans wanted longer terms than banks could grant or were new enterprises. For the sake of completeness, it should be pointed out that the realized net loss on defaulted loans reported above was in the one to two per cent range of the total amount disbursed on all loans. This simply indicates that the lending agency compensates for its estimating procedure errors by requiring sufficient collateral to recoup most of the amount loaned in case of default.

Having described the results of practices for estimating capital needs, it is time to focus on the practices themselves.

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[7] *Ibid.*, 47.

1.3. Techniques for Estimating Capital Needs

Among the many general admonitions to use caution in the standard literature on business finance, some procedures for estimating requirements for funds are available. In the opinion of the author, the best specific technique discovered for estimating initial capital requirements is that used in the Industrial (Small Business) Series of the Department of Commerce. Master worksheets are available for some two dozen different types of retail outlets.[8] As an example, the worksheet for establishing a shoe store consists of an itemized list of furniture, fixtures, and equipment; other nonrecurring items requiring initial capital; and the items to be estimated under monthly sales and operating expense. The form provides space for estimates beside each item, and the items listed are those found in the typical successful store of the designated type. The real value of the form is its presentation of the typical experience of successful stores in the industry by means of operating ratios (expense, margin, and profit items as a percentage of sales) along with a sample application of the ratios. In addition to knowing what the monthly rate of expense will be, the prospective businessman would like to know what income to expect during the first months and how long it will be before the firm will be paying its way. Here the form becomes conceptually weak. It requires the user to enter "one month or more" or "two months or more" for the various items of expense and makes no provision for nor establishes any basis for estimating income. In addition, no basis is given for the use of even the indefinite time periods specified for expense estimating. In point of fact, the operating ratios on which expense estimates are based are those of "going" concerns; they do not necessarily reflect experience during the *initiation* periods of similar firms. While the example above is from the field of retailing, essentially the same type of procedures (based on typical experience of factories of similar size in the industry) is recommended for manufacturers in one of the leading texts in the field.[9]

Another technique which is currently popular might be used for estimating the production rate required to equate cost and income rates. This is use of the "break-even chart."[10] Output rate is measured along the abscissa and money rate along the ordinate as is illustrated in Fig. 1.1. Line *A* is the revenue line. Line *B* is the total expense line. *P* is the break-even point. For output rates greater than 300 units per month net revenue will be positive. For example, at 400 units per month net revenue will be generated at the rate of $200 per month. If the total expense line is carefully calculated, the break-even chart does overcome one failing of the "going concern" comparison. It does show that the net cost rate below break-even is a decreasing function of output and, conversely, for net revenue above the break-even point. However, the essential ingredient for estimating the amount of initial funds required to meet early operating losses is missing. A relationship between demand rate or required output and elapsed time since commencing sales operations is needed. The nature of this function is critical to the determination of production policies to minimize cash needed or to maximize profits, for instance, as well as for determining the amount of cash required and the length of the period during which it will be needed. These latter quantities, of course, form the basis for estimating loan requirements.

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[8] A typical title is *Establishing and Operating a Retail Shoe Store*, Industrial (Small Business) Series No. 34, U. S. Department of Commerce, Superintendent of Documents, Washington, D. C.: U. S. Government Printing Office. A discussion of operating ratios, their sources and uses, and the worksheet can be found in P. C. Kelley and K. Lawyer, *How To Organize and Operate A Small Business*, Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1949, 530 ff. and Appendix B.

[9] P. C. Kelley and K. Lawyer, *Ibid.*, 139 ff.

On the other side of a formal loan transaction are bankers. Although operating ratios, other facets of typical experience for going concerns, and break-even analysis are available to bankers and presumably are used by some, the usual materials required as the basis for a loan are a formal application together with balance sheets, budgets, and operating statements. Loan applications require an inventory of other loans, the purpose for which the loan is being requested, and identification of source for repayment in addition to routine business and personal biographical data.[11] Another source which may be used for established firms is the credit rating.

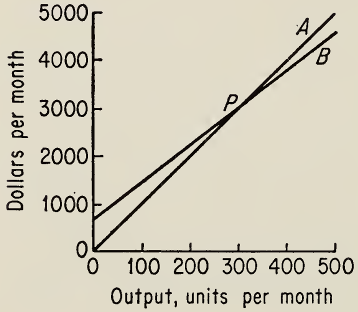

Fig. 1.1. Break-even chart

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[10] A full discussion of break-even charts can be found in Joel Dean, *op. cit.*, 326 ff.

[11] R. B. Tower, *A Handbook of Small Business Finance*, Small Business Management Series No. 15, Small Business Administration, Washington, D. C.: U. S. Government Printing Office, 1957, 31–39.

In brief, techniques available to estimate initial capital needs and the time for which they will be needed for new firms are composed of various aspects of static experience from established concerns in the relevant industry. No evidence has been found of attempts to build models of the time-dependent process or to establish quantitative estimates of the probability of success.

1.4. Purposes of this Study

The goal of this study is to establish some theoretical foundations which may eventually help persons concerned with starting new businesses solve five common problems:

1. How much initial operating capital is needed to cover net operating losses during the early, low-demand period?
2. How does production policy affect the initial operating capital requirement and how should production be scheduled to minimize that requirement?
3. When funds to cover initial operating losses are plentiful, how can production be scheduled to maximize profits over the initiation period?
4. How can initial demand experience of similar firms be combined with a given firm's financing, pricing, fixed costs, and variable costs, to determine that firm's probability of success?
5. What controllable factors influence a firm's probability of success and how should these be controlled to maximize that probability?

The techniques currently available for making the types of decisions described were criticized because of their dependence on *static* experience of *going* concerns. In contrast, this study concentrates specifically upon the initiation period and the flow through time of physical goods along with the counterflow of cash during this period. The cash flow is made the basis of techniques designed to estimate the initial operating capital requirement and the payout period required for any short-term loan made to cover part of initial expenses.

The meaning of *initial operating capital requirement* as used in this study can best be conveyed by tracing the net flow of cash from the time the owners decide to go ahead with founding a new firm. At first, money has to be paid out for plant, machinery, other equipment, and the services to assemble it all into an organized economic entity. This will be called *set-up expense.* Then production and sales operations are begun. However, the expense rate normally will exceed the revenue rate so expense will continue to mount until revenue rate becomes equal to expense rate. The difference between total net expense up to this point in time and set-up expense will be called *initial net operating expense.* An equal and opposite *initial operating capital requirement* covers this expense.

The *payout period* for a loan will be defined to begin at the commencement of operations rather than when the loan was actually made. Such a stipulation is made to establish a common point of reference that will be valid for all cases. The payout period is defined to end when cumulative net income from operations first becomes equal to the amount of the loan, regardless of the time *allowed* by the person lending the money.

In spite of the desirability of having it do so, the study cannot be read and applied directly by the person for whom it is intended — the businessman who wishes to appraise an opportunity for founding a new firm. In the first place, the work is concerned with investigating the mathematical structure of the process and with developing general models. Secondly, certain data needed to build up empirical probabilities from initiation period experience of a group of similar firms are not available. Nonetheless, it is felt that the study does contribute important understanding of the general characteristics of the solutions. In addition, some of the policy decisions depend on such general conditions that only a modicum of observation would be needed to tell what the decision should be. The data which are lacking are not unusual in concept; they are collected as a normal part of accounting processes by any well-run firm. However, they are not as yet collected, classified by industry, and published by any central source. Given the data, it is felt some of the results and techniques of this paper could be transformed into understandable and useful devices for businessmen, bankers, and those who advise prospective founders of new firms.

CHAPTER 2

Demand, Revenue, and Costs of Operation

2.1. Summary

Before a start can be made on the main tasks of the study, certain basic relationships are needed for demand, revenue, and costs of operation. The functional relationship between demand and time to be used throughout the study will first be defined and connected to revenue. Next, several types of operating costs will be taken up: production costs as a function of physical rate of output, the cost of transition from one level of production to another, costs associated with inventory, and the interest paid for the use of borrowed funds.

The principal result of the chapter is a proof that minimum production cost as a function of output rate is essentially linear for technologies composed of identical work stations; for example, for segmented technologies. Production costs as a function of output are built up sequentially. An illustrative case is presented to show how a production function can be built empirically for one machine. The form of this function is quite general and is used to construct a minimum production cost function when several machines are available. The result is perfectly general for identical machines and is probably a good approximation for any segmented technology. This result, however, is at variance with the shape of the production cost function traditionally assumed by economists. Empirical evidence is cited to support the shape of the function as derived.

2.2. Demand and Revenue

The prime determinant of a production schedule and of the principal costs of operation is the manner in which demand varies with time. Logical cases can be made for many parametric growth functions; for example, sigmoid or exponential growth. There is, however, no one function which has general acceptance. Consequently, a simple relationship between demand and time will be chosen. A simple representation that is in parametric form must have at least two parameters, one to control the rate of increase of demand and one to control the upper limit on growth.

Since this study will not attempt to establish a theory of demand, the simplest function which possesses the two features described in the previous paragraph will be used; namely, linear growth to a plateau:

$$d(t) = \begin{cases} ht & 0 < t < t_1 \\ q & t > t_1 \end{cases}$$

where d is demand in physical units per unit of time (t), h is a constant rate of growth which carries dimensions of units of product per month squared, and q is the ultimate constant number of units demanded per unit time. Certain results in Chapter 3, however, will be shown to hold for more general $d(t)$.

A relationship between revenue per unit time and demand will be required. A cash-and-carry or cash-on-delivery policy will be assumed such that

$$r(t) = p \cdot d(t)$$

where r is dollars of revenue per unit time and p is the price of a unit of product.

2.3. Production Cost

As an illustration of how a production cost-output rate function can be constructed from technological relationships, consider a firm operating a single plastic extruder. The extruder requires inputs of two types of resins and electric power. The electricity heats the resins and turns the internal extrusion screw which mixes the molten resins and forces the plastic film through the die at the output end where it is taken up on an electrically powered roller. The input resins must enter in fixed proportions and their combined weight is equal to the weight of the product within the operating range of the extruder.

Let x_1 and x_2 be the input rates of the resins in pounds per hour and let x_3 be the input of electricity in kilowatts. The production rate will be y units of product per operating hour. The machine's rated capacity (s) in pounds per hour will be defined as the production rate at which the marginal consumption of electricity (dx_3/dy), in terms of kilowatt hours per pound, is a minimum. The following relationship is assumed.

$$(2.1) \qquad (y - s)^2 = v\left(\frac{dx_3}{dy} - u\right) \qquad s > 0, \qquad u > 0, \qquad v > 0$$

Thus u is the minimum marginal consumption of electricity. It is presumed that the numerical values of s, u, and v are obtainable either from the manufacturer or from other users of the same type of machine. The conversion factor v has dimensions of

$$\frac{\text{pounds}^3}{(\text{kilowatt})(\text{hour})^3}$$

It should be noted that the quadratic form in equation (2.1) represents the first three terms of a Taylor expansion of a general polynomial approximation to the relationship between output and marginal consumption of electricity. Finally, it is presumed that the costs per pound of the input resins, α and β, as well as the cost of a kilowatt hour of electricity, λ, are known.

Solution of (2.1) for x_3 yields

$$(2.2) \qquad x_3 = \frac{ay^3}{3} - \frac{by^2}{2} + cy$$

where

$$a \equiv \frac{1}{v}, \qquad \frac{b}{2} = \frac{s}{v}, \qquad \text{and} \qquad c \equiv \frac{s^2}{v} + u$$

The expression $ay^2 - by + c$ must be positive for all y, and a must be positive since the marginal consumption of electricity must be a positive convex function of output rate. Parenthetically, the discriminant ($b^2 - 4ac$) must be negative.

The relationships between the physical quantities of inputs and outputs for the machine can be summarized in three technological equations which are most easily expressed as follows:

$$y = kx_1$$

$$x_1 = ex_2$$

$$x_3 = \frac{a}{3}y^3 - \frac{b}{2}y^2 + cy$$

where k and e are dimensionless proportionality constants.

Now assume that production costs per unit time z are related to the three inputs as follows:

$$z = \alpha x_1 + \beta x_2 + \lambda x_3 + F$$

where the coefficients are the unit costs of the factors as previously defined. F is the fixed cost rate per unit time and includes normal overhead plus the cost of labor. The production cost-output rate function can then be found by substituting the relationships expressed in the technological equations.

$$z = \lambda\left(\frac{ay^3}{3} - \frac{by^2}{2} + cy\right) + \left(\frac{\beta + \alpha e}{ke}\right)y + F \tag{2.3}$$

in which z will increase monotonically as y increases.

Some checks can be made to see if (2.3) behaves in accordance with economic principles. The average and marginal cost functions are

$$\bar{z} = \frac{z}{y} = \lambda\left(\frac{ay^2}{3} - \frac{by}{2} + c\right) + \frac{\beta + \alpha e}{ke} + \frac{F}{y} \tag{2.4}$$

$$z' = \frac{dz}{dy} = \lambda(ay^2 - by + c) + \frac{\beta + \alpha e}{ke}$$

The form of (2.4) shows that it is asymptotic to the z-axis as y goes to zero, which is as would be expected. The marginal cost function is a parabola which is convex and which has a minimum greater than zero because

$$\min\,[z'] = \frac{\lambda}{4a}(4ac - b^2) + \frac{\beta + \alpha e}{ke}$$

and $b^2 - 4ac < 0$. Furthermore, $\bar{z}$ (average cost) is a minimum when

$$\lambda y^2\left(\frac{2}{3}ay - \frac{b}{2}\right) = F$$

and $\bar{z} = z'$ when

$$\lambda y^2\left(\frac{2}{3}ay - \frac{b}{2}\right) = F$$

Therefore the marginal cost curve intersects the average cost curve at its minimum value which is typical of such functions. Figure 2.1 illustrates these relationships when $\alpha = 7$, $\beta = 9$, $\lambda = 11$, $F = \$500$, and when a, b, and c take the values $\frac{1}{2}$, 4, and 13.

The general production cost curve of which equation (2.3) is a special case will now be taken up.

$$z = Ay^3 - By^2 + Cy + F \tag{2.5}$$

The following restrictions are required on the constants in order to have z physically meaningful

$$A > 0, \qquad B > 0, \qquad F > 0, \qquad B^2 - 3AC < 0$$

When the function is built up from technological data, these restrictions would arise from the basic physical relationships used to build the equation.

Most factories have more than one machine. Therefore, the single machine cost function for the general case (2.5) will be used to build a cost function for a factory with many machines. To do this requires solving an optimal allocation problem, since a company should distribute the production over the machines to achieve least cost.

Before this is done it might be well to note that the historic pattern in American industry has been to adjust for errors in forecasting and planning primarily by varying the production rate of the firm. Faced with uncertain-

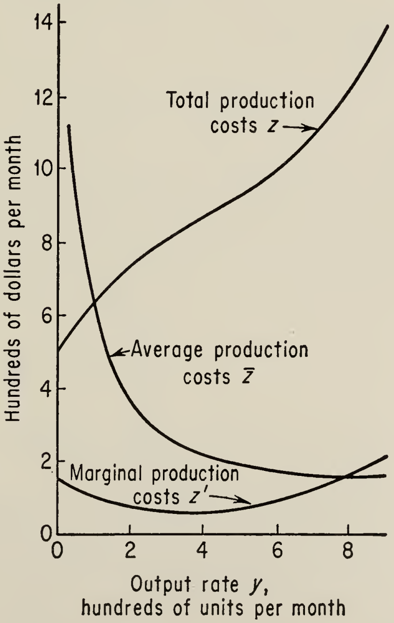

Fig. 2.1. Total, average, and marginal production costs as a function of output rate

ties in estimating the future it would be no great surprise to discover that American management has found ways to incorporate flexibility into "fixed" plant in order to adapt to rather wide swings in desired output rate without materially affecting unit costs.

In equation (2.5) fixed costs F are assumed to be a characteristic of the plant as a whole rather than of the number of work stations operating. Furthermore, the portion of the total factory production load y assigned to the ith working station is defined to be x_i. Given these conditions, it is asserted that, if the load y is such that the optimal number of working stations ($x_i > 0$) is n, then the optimal load for each station is

$$x_i = \frac{y}{n}$$

To prove the assertion let $f_n(y)$ be the cost of producing y by optimally loading n stations. Then, when $n = 1$, by necessity $x = y$ and

$$f_1(y) = Ay^3 - By^2 + Cy + F$$

when $n = 2$,

$$f_2(y) = \min_{0 \leq x \leq y} [Ax^3 - Bx^2 + Cx + f_1(y - x)]$$

The minimum might occur at an end point; that is, when $x = 0$ or $x = y$, but, if so, one station will be idle. This is contrary to the hypothesis that y is in the range where the optimal number is $n = 2$. Substituting for $f_1(y - x)$ and calling the unminimized production cost Z leads to

$$\frac{dZ}{dx} = (2x - y)(3Ay - 2B)$$

$$\frac{d^2Z}{dx^2} = 2(3Ay - 2B)$$

so that Z is a minimum for $x = (y/2)$ if $y > (2B/3A)$.

If $y < (2B/3A)$, Z has a maximum at $x = (y/2)$. Furthermore, Z decreases monotonically by decreasing or increasing x. The smallest value of Z occurs at $x = 0$ or, equivalently, at $x = y$. Both of these solutions involve an idle station so that $y < (2B/3A)$ must be outside the considered range of y. Hence, if two stations working is optimal, each must have half the load to minimize Z.

Proceeding now by induction, suppose that the assertion is true for $(n - 1)$ stations. Then

$$f_{n-1}(y) = \frac{Ay^3}{(n-1)^2} - \frac{By^2}{n-1} + Cy + F$$

and

$$f_n(y) = \min_{0 \leq x \leq y} [Ax^3 - Bx^2 + Cx + f_{n-1}(y - x)] \tag{2.6}$$

Note that if y is such that n stations working is optimal, the minimizing value of x in (2.6) gives a value of $(y - x)$ which must optimally require $(n - 1)$ working. Consequently, the expression above for f_{n-1} may be used in (2.6).

Now,

$$\frac{dZ}{dx} = \frac{n}{(n-1)^2}\left[x - \frac{y}{n}\right][3A(n-2)x + 3Ay - 2(n-1)B]$$

$$\frac{d^2Z}{dx^2} = \frac{n}{(n-1)^2}\left[6A(n-2)x + \frac{6Ay}{n} - 2(n-1)B\right]$$

The first derivative is zero for

$$x_1 = \frac{y}{n}$$

$$x_2 = \frac{2B(n-1)}{3A(n-2)} - \frac{y}{n-2}$$

and the corresponding second derivatives are

$$\left.\frac{d^2Z}{dx^2}\right|_{x_1} = \frac{6A}{n-1}\left(y - \frac{nB}{3A}\right)$$

$$\left.\frac{d^2Z}{dx^2}\right|_{x_2} = \frac{6A}{n-1}\left(\frac{nB}{3A} - y\right)$$

Therefore, for $y > (nB/3A)$, $x_1 = (y/n)$ minimizes Z and x_2 maximizes Z, which is as required for the induction. In contrast, it can be shown that, for $y < (nB/3A)$, the optimal number of stations working is not n. For $y < (nB/3A)$ there must be a pair of stations with a combined load y' such that

$$y' < \frac{2B}{3A}$$

since the average combined load for station pairs with n stations working is less than $2B/3A$. From the case wherein $n = 2$, it is known that one of the two stations should be idle. Hence, for $y < (nB/3A)$, n is not the optimal number. In conclusion then, it has been proved by induction that for loads such that the optimal number of stations working is n, the optimal loading for the plant is achieved by distributing the load equally over the n machines.

If n is the number of stations producing and if $y > (nB/3A)$, the minimum total cost relationship is

$$z = \frac{Ay^3}{n^2} - \frac{By^2}{n} + Cy + F \tag{2.7}$$

This equation expresses minimum total cost for a given n. There is still the problem of determining when to shift from n stations to $n + 1$ stations. This can be done by means of the following recursion.

$$\frac{Ay^3}{n^2} - \frac{By^2}{n} + Cy + F = \frac{Ay^3}{(n+1)^2} - \frac{By^2}{(n+1)} + Cy + F$$

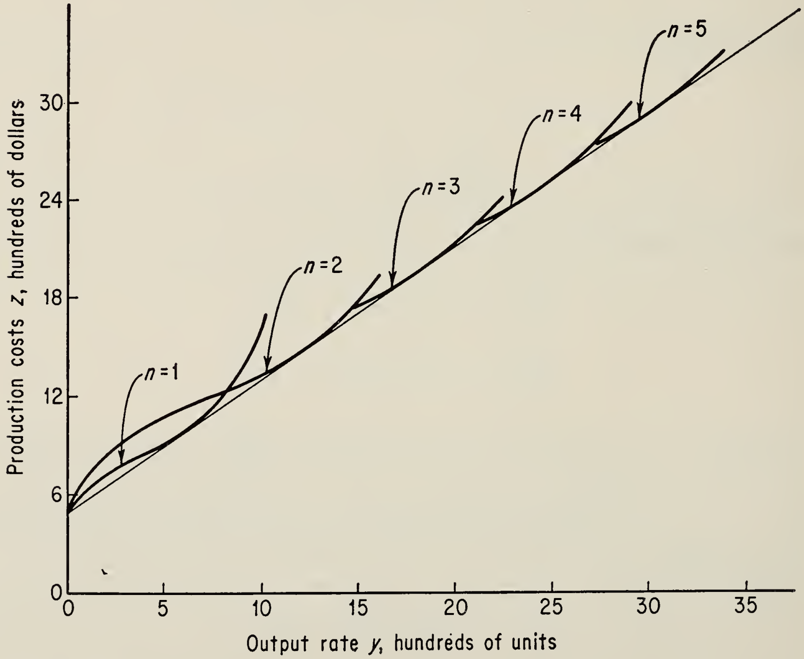

Fig. 2.2. Production costs as a function of output rate and number of machines producing

from which

$$y = \frac{B}{A} \cdot \frac{n(n+1)}{2n+1}$$

Thus the rule is to operate n stations when y is in the range

$$\frac{B}{A} \cdot \frac{n(n+1)}{2n+1} > y > \frac{B}{A} \cdot \frac{n(n-1)}{2n-1}$$

Note that y satisfies $y > n(B/3A)$ as required for $x_i = y/n$ to give minimum cost.

The resulting total cost curve has a scalloped shape and is illustrated in Fig. 2.2. The numerical values used to compute the functions are the same as those used in Fig. 2.1.

Of particular interest is the tangent to the cost curve; not only because it is a good approximation to the curve (especially for large n) but also because it turns out to be linear. Since the scallops are caused by the discrete nature of the stations and would disappear if fractional stations could be had, a simple way to find the equation for the tangent is to make n continuous.

Selecting n to minimize cost by differentiating (2.7) leads to

$$n = \frac{2Ay}{B}$$

and

$$z = \left(C - \frac{B^2}{4A}\right)y + F$$

which is the equation for the tangent and a good approximation to the cost function for a segmented technology. Hence average costs and marginal costs are

$$\bar{z} = C - \frac{B^2}{4A} + \frac{F}{y}$$

and

$$z' = C - \frac{B^2}{4A}$$

For large output rates average cost approaches marginal cost and marginal cost approaches a constant.

The essential linearity of the n-machine total cost curve in the model above is widely at variance with the assumption made by economists in a preponderant majority of their theory about the individual firm. The classic assumption is that the marginal cost function is "U-shaped," that it is a positive convex function of output. Indeed this assumption is necessary to support the general equilibrium conclusions which make up a large part of micro-economic theory. Faced with this contradiction one is led to see whether there is any empirical evidence which might help resolve the issue.

Joel Dean has conducted a number of empirical studies to determine the shapes of cost-output functions, or total cost curves, in actual factories that change output rate by adding or subtracting homogeneous production units such as are found in the above hypothetical factory. In describing his techniques he reports that,

> To reach a choice concerning the shape of the total cost function, three investigations were made: First, the process of manufacturing was examined in the light of the theoretical analysis of segmentation and variable proportions presented earlier. Second, the statistical distribution of cost observations was examined by graphic multiple regression analysis. This usually included a study of the scatter of first differences of cost on output; . . . the linear relationship of first differences . . . combined with the lack of any evidence of rising per unit cost at extreme output . . . substantiated the hypothesis that the relationship of total cost was linear. A third step was to fit not only a straight line but curves.[1]

After which he states,

> In none of the plants studied did the first difference tests or the tests of significance for the higher degree functions indicate that a curvilinear total-cost curve fitted the data better than a straight line.[2]

The plants studied were a leather belt shop, a hosiery mill, and a furniture factory.

Dean's theoretical arguments on segmentation are qualitative. He says,

> The characteristics of the fixed equipment play a dominant role in the determination of a plant's pattern of short-run cost

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[1] Joel Dean, *op. cit.*, 291.

[2] *Ibid.*, 292.

> behavior. . . . The critical characteristic for this purpose is the degree of segmentation possible in the plant, that is, its potentiality of varying the rate of flow of output without changing the proportions of variable inputs to fixed equipment in use.[3]

Dean discusses three types of segmentation: unit segmentation — varying the number of homogeneous production units in use; time segmentation — varying the number of hours or shifts; and speed segmentation — varying the rate of the assembly line.

Another investigator, who has applied regression analysis to costs in British industry, is Philip Lyle. He says,

> We have found that for sugar refineries . . . the total cost curve is a straight line with a positive gradient.[4]

Finally, one of the country's noted economists summed up the situation with respect to individual firm cost functions this way:

> Another important source of knowledge of the cost of the enterprise . . . has been the series of statistical studies of costs which have appeared since 1933. Statistically derived cost curves designed to show the relation between variations in output and cost per unit of product have been constructed from accounting data for a considerable variety of industries. Statistical procedures have been employed for the elimination of the effects upon cost of influences other than variations in output, such as changes in the prices of factors, technological changes, and the lag of output behind costs; and the residual relationship between total cost and output has then been measured by a line of regression of best fit [sic]. In most cases the resulting cost curve has been representative, of necessity, of the behavior of costs for the aggregate of products of the concern rather than for a single, undifferentiated commodity.
>
> With only very few exceptions the line . . . of best fit has been a straight line, and consequently the conclusion has been drawn that marginal cost within the range variation of output covered . . . has probably been constant. Such a conclusion is of course

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[3] *Ibid.*, 274.

[4] Philip Lyle, *Regression Analysis of Production Costs and Factory Operations*, London: Oliver and Boyd, Ltd., 1946, 107.

> widely at variance with the economists' U-shaped marginal cost curve for the individual concern.[5]

It would seem that there now is both empirical and theoretical evidence to support the case for an essentially linear production cost function for firms characterized by segmented technologies. When one thinks of machine shops, assembly shops, small forging plants, handicraft industries, etc., which vary production by adding or subtracting virtually homogeneous work stations, it becomes evident that segmentation is a feature of an important portion of industry.

2.4. Transition Cost

In this study transition costs are defined as those costs resulting from changing the production rate. Where transition costs are considered, production changes will be assumed to take place in discrete steps, each accompanied by a cost independent of the size of the step. It is assumed that each transition costs T, where T is a constant. Many real situations will be more complicated than this, but this case should help make clear the main features of policies required in the face of substantial transition costs.

An example of the sort of technology which has transition costs of the sort described might be a highly integrated continuous-flow chemical plant in which the adjustment to a different output rate requires that a crew be called out to watch gauges and adjust subsidiary valves and governors as the main input stream valves and pumps are speeded up or slowed down to the new rate. A petroleum refinery and a mass production assembly line are other examples. The characteristics of interest are (1) that a change in output rate requires a delicate balancing of flows of raw and semi-finished materials, and (2), once the balancing adjustments are made, the new output rate established can be maintained by relatively simple controls.

2.5. Inventory and Inventory Cost

Unless specifically stated otherwise, the assumption in all models is that no shortage is permitted. Shortages do occur in practice, but it would not appear that they play an important part in starting a firm (at least intentionally) and so this complication has been suppressed to keep the models down to essentials.

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[5] Bernard F. Haley, "Value and Distribution," *A Survey of Contemporary Economics*, H. S. Ellis (ed.), Philadelphia: The Blakiston Co., 1948, 13.

Inventory is defined to be

$$I(t) = \int_0^t [y(x) - d(x)]dx$$

where $d(x)$ is the demand function, $y(x)$ is the physical production function, and t is time measured from the commencement of operations.

Inventory costs per unit time are assumed to be proportional to the amount in inventory. They include only handling and storage costs and, hence, can be expected to be small as compared with production costs for the quantity of goods in inventory. They do not include opportunity cost on the value of the goods in inventory because these costs are included in the loan costs on capital required for the entire business, including inventories. Symbolically, inventory costs are

$$K(t) = s \int_0^t I(x)dx$$

where s is the cost rate in terms of

$$\frac{\text{dollars}}{\text{(pound)(month)}}$$

for example.

2.6. Loan Cost

One type of loan arrangement is considered — the discounted loan. No reduction in interest charged will be made for repayment of the loan prior to the expiration of the agreed term. Use of this type of loan makes explicit the typical feature of practically all business loans — the amount required to pay interest on the loan comprises part of the loan itself. In practice, the effective rate of interest on such a loan could be reduced by investing any funds intended for repayment until the term of the loan had expired. However, this refinement will be ignored.

Consider a firm that needs, in addition to its equity capital, an amount W_0 to complete the setting up of plant and other preparations and an amount W^* to cover the net operating loss up to the break-even point. The loan L required will be

$$L = \frac{W^* + W_0}{e^{-it_L}}$$

where i is the interest rate and t_L is the contractual term of the loan.

It should be pointed out that W_0 could also be a negative quantity. It would then be the amount remaining out of initial equity after set-up has been completed.

CHAPTER 3

Production Decisions When There Are No Transition Costs

3.1. Summary

Scarcity of capital is a prevalent condition among persons who want to start small firms. Consequently, minimization of initial operating capital requirements is a realistic objective for such cases. Another objective might be found among owners of established parent firms who wish to start branch plants or subsidiaries. In this second class of cases there is likely to be no capital restriction to prevent operating so as to maximize profits during the initiation period.

Three different technologies, two in this chapter and one in the next, will be treated under each of the two criteria. The first technology has a linear production cost function and can change output rate continuously without transition cost. The second has no transition cost but has a nonlinear production cost function. The third has linear production costs and an appreciable transition cost is associated with any change, no matter how small, in output rate. A technology with nonlinear production costs and appreciable transition costs constitutes a fourth type. However, it has not been possible to develop any useful general solutions for this type. Consequently, it has not been included.

To summarize the principal results of the chapter, it is shown that producing to demand is optimal for both minimizing operating capital requirements and maximizing profits with respect to the first technology.

The proof is based on a perfectly general demand function; that is, it is not restricted to the linear growth demand function described in Chapter 2. In the case of the second technology, the concept of time segmentation is given analytic meaning and is used both alone and in conjunction with unit segmentation to produce a linear production cost function. Thus the optimal production policy is the same as in the first case. Operating capital requirements and loan payout periods are calculated for firms having either of these first two technologies.

3.2. Definitions

Net initial operating expense $W(t)$ is defined to be

$$W(t) = Tn(t) + Z(t) + K(t) - R(t)$$

where T is the cost of a transition, $n(t)$ is the cumulative number of transitions to t, $Z(t)$ are cumulative production costs, $K(t)$ are cumulative inventory costs, and $R(t)$ is cumulative revenue. Time is measured from the commencement of production operations. Pro rata loan costs are excluded from $W(t)$ because it is visualized as a net cash flow function. As described in Chapter II, the loan was paid for at the time it was made and repayment of the principal will not occur until the end of the term. The funds that will be used to pay the loan will come from net revenue after the break-even point (defined below) has been passed. Hence no expense against the sum of loan proceeds and W_0 will occur during the initiation period.

The *net operating capital requirement* (W^*) is max $[W(t)]$. When operations are begun there will be no revenue, but there will be fixed and variable costs of production, at least. Hence net expenses will cumulate until, at *break-even time* (t'),

$$r(t') = z(t') + k(t')$$

where the small letters represent the rates of the corresponding capital-letter cumulants mentioned above. Beyond t', revenue rate will exceed cost rate and $W(t)$ will decline except for possible transition costs.

With regard to any loan which has been made, the total amount that will have to be repaid to the bank is L even though only the discounted proceeds are available for use. The *payout time* or period on this loan is defined to be the smallest t_3 for which all $t \geq t_3$ satisfy

$$M - W(t) \geq L$$

where M is the amount available to cover expenses at $t = 0$. In this chapter, with known demand, it is sensible to let $M = W^*$.

Although $\theta(t) = -W(t)$ could be defined as a profit function, no profit function is required to pursue the objective of maximizing initiation-period profits. Since no shortage is permitted, the physical demand function and, consequently, the revenue function remain unchanged by variations in production policy. Therefore, minimization of relevant costs accomplishes the same objective.

3.3. Linear Production Cost

With linear production cost and no transition cost, it will be shown that the optimal production policy for minimizing initial operating capital requirements and for maximizing profit is to produce to demand. The initial operating capital requirement (W^*) and the payout period (t_3) will be found for this case.

Consider a linear production cost function $z = a + by$ where, for example,

$$b = C - \frac{B^2}{4A}$$

as in Chapter 2. Also consider a general demand function $d(t)$ which eventually reaches an ultimate rate q at which

$$pq > a + bq$$

where p is price.

The no-shortage restriction requires that

$$Y_i(t) \geq D(t)$$

where $Y_i(t)$ is cumulative physical production up to time t under production policy i and $D(t)$ is cumulative demand up to t. The *net expense rate* is, in general,

$$w_i = a + b[y_i(t)] - p[d(t)]$$

where $y_i(t)$ is the production function corresponding to $Y_i(t)$ and where inventory costs are omitted for the present.

Let subscript 0 refer to a policy of producing to demand. Then

$$W_0(t) = at + b[D(t)] - p[D(t)]$$

Now $W_0^* = \max [W_0]$ will occur at some t'_0 such that

$$W_0^* = at'_0 + b[D(t'_0)] - p[D(t'_0)]$$

This maximum exists provided $a > 0$ and $pq > a + bq$.

Next, assume that $y_1(t)$ is a production function which is at least as good as $y_0(t)$. Define W_1^* as occurring at t'_1. Then

$$W_1(t) = at + b[Y_1(t)] - p[D(t)]$$
$$W_1^* = at'_1 + b[Y(t'_1)] - p[D(t'_1)]$$

and

$$W_1(t'_0) = at'_0 + b[Y_1(t'_0)] - p[D(t'_0)]$$

In order for the production policy $y_1(t)$ to be at least as good as $y_0(t)$,

$$W_0^* \geq W_1^* \geq W_1(t'_0)$$

The first and last terms of this continued inequation state that

$$at'_0 + b[D(t'_0)] - p[D(t'_0)] \geq at'_0 + b[Y_1(t'_0)] - p[D(t'_0)]$$

or

$$D(t'_0) \geq Y_1(t'_0)$$

Hence, with respect to production costs, $y_1(t)$ can be just as good as producing to demand but it cannot be better than producing to demand because, in order for it to be better, it must violate the no-shortage restriction. Now, if $y_1(t)$ is not synonymous with producing to demand prior to t'_0, then at some previous point in time there had to be some inventory generated. The cost of handling and storing this inventory will make producing to demand preferable even though cumulative *production* costs under *all* policies are the same up to any point in time at which there is no finished inventory on hand. The conclusion is that, for this case, producing to demand will minimize the initial operating capital requirement.

For all t, cumulative costs will be minimized by producing to demand. Since the revenue function (whatever it may be) is unaffected by the production policy, producing to demand is also the correct policy for maximizing the profits over the initiation period.

It seems rather striking that for segmented technologies which have negligible transition costs the same production policy permits realization of what might at first appear to be two conflicting objectives. In addition, one could have no simpler forecasting problem than that created by the objective of producing to short-run demand. The less the lead time which must be built into a forecasting procedure, the more accurate the predictor. Certainly no forecasting procedure could be more accurate than when predicting needs for just the next planning period.

If the physical demand function is the two-part linear function described in the early part of Chapter 2, t' and W^* can be calculated readily. For any profitable cash business,

$$pq > a + bq \quad \text{or} \quad q > \frac{a}{p-b}$$

This means that the revenue rate must equal expense rate prior to t_1, that is, break-even time (t') must occur before ultimate demand (q) is reached, or

$$pht' = a + bht'$$

so

$$t' = \frac{a}{h(p-b)}$$

Then

$$W_0^* = W_0(t') = at' + b\frac{h(t')^2}{2} - p\frac{h(t')^2}{2}$$

$$W_0^* = \frac{a^2}{2h(p-b)}$$

Implicitly, the payout period t_3 for a loan of size L, if demand continues to grow at rate h, satisfies

$$L = \frac{ht_3^2}{2}(p-b) - at_3 + \frac{a^2}{2h(p-b)}$$

or explicitly,

$$t_3 = \frac{a}{h(p-b)} + \sqrt{\frac{2L}{h(p-b)}} = t' + \sqrt{\frac{2L}{h(p-b)}}$$

Thus, if L is precisely W_0^*,

$$t_3 = \frac{2a}{h(p-b)}$$

If demand ceases to grow at $t_1 < t_3$ (where $t_1 = q/h$),

$$L = [(p - b)q - a]\left[t_3 - \frac{q}{h}\right] + \frac{q^2}{2h}(p - b) - \frac{aq}{h} + \frac{a^2}{2h(p - b)}$$

or

$$t_3 = \frac{L + \dfrac{q^2(p - b)}{2h} - \dfrac{a^2}{2h(p - b)}}{(p - b)q - a}$$

If $L = W_0^*$,

$$t_3 = \frac{q^2(p - b)}{2h[(p - b)q - a]}$$

3.4. Nonlinear Production Cost

In this section it is shown that, when there are no transition costs, the device of time segmentation can be used to make certain nonlinear production costs effectively linear. Then the results of the previous section can be used for the case considered here.

Suppose a plant has the following production cost function.

$$z(y) = Ay^3 - By^2 + Cy + f$$

in dollars per hour where y is output rate per hour and A, B, C, and f are constants. If there are a hours in a standard production day, then $F = fa$ will be the fixed cost per day regardless of the output rate. Thus, only the variable costs can be controlled by output rate.

Suppose that the demand is u for a given day. Then u/y is the number of hours run required on this day if a production rate y is chosen. Total variable costs v will be

$$v = (Ay^3 - By^2 + Cy)\frac{u}{y}$$

or

$$v = (Ay^2 - By + C)u$$

Choosing y to minimize v yields a best output rate of

$$y_0 = \frac{B}{2A}$$

which, in turn, gives

$$\min\,[v] = \left[C - \frac{B^2}{4A}\right]u$$

and

$$z(y_0) = \left(C - \frac{B^2}{4A}\right)u + F$$

for daily production costs.

The restriction on achieving linearity for the production cost function by time segmentation is that $u \leq ay_0$ in the above. However, suppose there are several machines available as in Chapter 2, then

$$v = \left[\frac{Ay^2}{n^2} - \frac{By}{n} + C\right]u \quad \text{if} \quad \frac{Bn(n+1)}{A(2n-1)} > y > \frac{Bn(n-1)}{A(2n-1)}$$

and n is the smallest number of machines for which

$$u \leq \alpha\left(\frac{nB}{2A}\right)$$

The value of y which minimizes cost is

$$y_0 = n\frac{B}{2A}$$

which satisfies the restrictions on y.

Hence,

$$z = \left[C - \frac{B^2}{4A}\right]u + F$$

throughout the range of combined operation of the machines. Thus, for production processes in which a cubic will fit the total cost-output data of the individual machine adequately, and the assumptions involved in time and unit segmentation apply, segmentation can be used to establish an effectively linear production cost function.

Linearity conditions will apply from zero output up to the point at which all machines are operating beyond their individual minimum average cost output rates. If the plant is designed so that the ultimate expected demand rate q is somewhat below this output rate, the results of the previous section can be used by setting $b = C - (B^2/4A)$. Producing to demand will, once again, minimize initial operating capital requirements and will also minimize initiation period expenses.

CHAPTER 4

Production Decisions With Nonzero Transition Costs

4.1 Summary

Under assumptions that production costs are linear with output rate, that no shortage is permitted, and that a change in production rate of any amount is accompanied by a constant transition cost of T, two problems are solved in this chapter. The first problem is that of finding a production schedule that minimizes the initial operating capital requirement. The second consists of determining the production schedule that maximizes profits over the initiation period under the further assumption that at least enough money is available to meet the initial operating capital requirement.

In connection with the first problem, it is shown that there is an optimal policy characterized by zero inventories at production rate transition points and equal net expense peaks between transition times except for the final peak which may be lower. Each of the peaks, with the possible exception of the last is, therefore, equal to the minimum operating capital requirement.

The solution to the second problem also is characterized by zero transition inventories. Another characteristic is that the period up to the time at which demand reaches its plateau is divided into equal time intervals each with its appropriate production rate. The number of such intervals is determined so as to maximize profits.

4.2 Minimizing Initial Operating Capital Requirements

In the model considered, inventory storage and handling costs will be ignored. These costs usually will be on the order of two or three per cent of the production cost of the goods in inventory. Furthermore, the proof will show that inventory drops to nothing at transition points.

It should be pointed out, also, that the problem considered here is most naturally conceptualized in a dynamic programming framework, but attempts to solve it in that framework were unsuccessful. Without recourse to dynamic programming, however, *an* optimal policy can be found.

Two definitions are necessary. First, because a transition cost of T is associated with any amount of change in production rate, a *production policy* is defined to be a series of level production rates with transitions between. Second, an *optimal production policy* is defined to be one which minimizes the initial operating capital requirement.

An optimal production policy will be found by the following procedure:

1.1. A series of steps will be given by which any policy can be modified without increasing the peak capital requirement into one of a special class of policies.

1.2. A unique policy in this class will be identified and shown to be optimal; a procedure for finding this optimal policy will be described.

1.1. Steps Tending to Improve a Policy. Given any policy one can be found which is at least as good and in which:

1.11. There is no inventory at transition points.

1.12. There is at most one transition for $t \geq t_1$ where t_1 is the time at which demand reaches its plateau.

1.13. There is at least one transition for $t \geq t_1$.

1.14. All peaks in the policy are B peaks and are equal except for the last which is an A peak and may be lower. (A and B peaks will be defined subsequently.)

1.11. *No Inventory at Transitions.* Given any policy, there is a policy at least as good with no inventory at transitions.

There is no inventory at the time of the first transition; that is, at

$x_1 = 0$. Since the first transition occurs at $t = 0$ when operations are begun, it is obvious that there is no inventory.

Consider a policy with inventory at the last transition: $I(x_f) > 0$. Since the last transition must be to the ultimate demand and production rate q, the result would be a permanent inventory greater than zero. However, it is possible to postpone the time of the last transition until $I(x_f) = 0$. This cannot affect revenue and it postpones an increase in cost rate as well as the transition cost itself. Thus net expense as a function of time $W(t)$ for all $t \geq x_f$ will be decreased by the production cost of the excess inventory. Furthermore, since there will be a peak in the net expense function at x_f and since, in any profitable business, the ultimate revenue rate exceeds the cost rate associated with any level of output equal to or less than q, the peak at x_f can be reduced by postponing the final transition until $I = 0$.

Consider a policy wherein inventory is greater than zero at an intermediate transition point. Consider any three consecutive transition points x_i, x_{i+1}, x_{i+2} with $I(x_{i+1}) > 0$ and let

$$m_i = \text{production rate for } t \text{ such that } x_i < t < x_{i+1}$$

$$m_{i+1} = \text{production rate for } t \text{ such that } x_{i+1} < t < x_{i+2}$$

It is asserted that production rates m'_i and m'_{i+1} can be chosen such that $I(x_{i+1}) = 0$ and that this new policy is as good as or better than the previous policy.

A general relationship will first be established. By definition,

$$W(t) = Z(t) - R(t) + Tn(t)$$

where T is the cost of a transition, $n(t)$ is the cumulative number of transitions to t, $Z(t)$ is cumulative production cost, $R(t)$ is cumulative revenue. The assumptions previously stated mean that

$$W(t) = at + bY(t) - pD(t) + Tn(t)$$

where a and b are fixed and variable cost rates respectively, $Y(t)$ is cumulative output, $D(t)$ is cumulative demand. Now

$$I(t) = Y(t) - D(t) \geq 0$$

hence

$$W(t) = at - (p - b)D(t) + bI(t) + Tn(t)$$

Within the restrictions imposed, no change in production policy can affect the first two terms on the right-hand side of this last equation and, if all transition times are held constant, $I(t)$ is the only controllable quantity affecting $W(t)$. Thus a policy which decreases $I(t)$ in an interval and leaves it unaffected elsewhere can only lower any net expense peaks in that interval.

Consider $I(t)$ for $x_i \leq t \leq x_{i+2}$.

$$I(t) = I(x_i) + m_i(t - x_i) - [D(t) - D(x_i)], \qquad x_i \leq t \leq x_{i+1}$$
$$I(t) = I(x_{i+1}) + m_{i+1}(t - x_{i+1}) - [D(t) - D(x_{i+1})], \qquad x_{i+1} \leq t \leq x_{i+2}$$

The two production rates can be determined from

$$I(x_i) + m_i(x_{i+1} - x_i) - [D(x_{i+1}) - D(x_i)] = I(x_{i+1}) > 0$$
$$I(x_{i+1}) + m_{i+1}(x_{i+2} - x_{i+1}) - [D(x_{i+2}) - D(x_{i+1})] = I(x_{i+2})$$

However, inventory at x_{i+1} can be made zero if m_i and m_{i+1} are chosen such that

$$I(x_i) + m'_i(x_{i+1} - x_i) - [D(x_{i+1}) - D(x_i)] = 0$$
$$0 + m'_{i+1}(x_{i+2} - x_{i+1}) - [D(x_{i+2}) - D(x_{i+1})] = I(x_{i+2})$$

from which

$$(m'_i - m_i)(x_{i+1} - x_i) = -I(x_{i+1})$$
$$(m'_{i+1} - m_{i+1})(x_{i+2} - x_{i+1}) = +I(x_{i+1})$$

or

$$m'_i = m_i - \frac{I(x_{i+1})}{x_{i+1} - x_i}$$
$$m'_{i+1} = m_{i+1} + \frac{I(x_{i+1})}{x_{i+2} - x_{i+1}}$$

Under this policy

$$I'(t) = I(x_i) + m'_i(t - x_i) - [D(t) - D(x_i)], \qquad x_i \leq t \leq x_{i+1}$$
$$I'(t) = m'_{i+1}(t - x_{i+1}) - [D(t) - D(x_{i+1})], \qquad x_{i+1} \leq t \leq x_{i+2}$$

Therefore,

$$I(t) - I'(t) = (m_i - m'_i)(t - x_i) = I(x_{i+1})\,\frac{t - x_i}{x_{i+1} - x_i} \geq 0, \quad x_i \leq t \leq x_{i+1}$$

$$\begin{aligned} I(t) - I'(t) &= (m_{i+1} - m'_{i+1})(t - x_{i+1}) + I(x_{i+1}) \\ &= I(x_{i+1})\left[1 - \frac{t - x_{i+1}}{x_{i+2} - x_{i+1}}\right] \geq 0, \qquad x_{i+1} \leq t \leq x_{i+2} \end{aligned}$$

Hence the policy with $I(x_{i+1}) = 0$ is at least as good as any one with

$$I(x_{i+1}) > 0$$

One point remains to be demonstrated. In the expression

$$m'_i = m_i - \frac{I(x_{i+1})}{x_{i+1} - x_i}$$

m'_i must be greater than or equal to zero. That this can always be made to be the case can be shown by considering the first two production rates. Since $I(x_1) = 0$ when $x_1 = 0$ and since m_1x_2 will be greater than $I(x_2)$ by the amount sold up to x_2, there will be no violation of this restriction in setting $I(x_2)$ equal to zero. Now, for the second rate, $I = 0$ again at the start and the same argument applies. Proceeding on sequentially, there will be no violation elsewhere.

The conclusion is that inventories can be made equal to zero at all transitions without increasing peak capital requirements.

1.12. *At Most One Transition For* $t \geq t_1$. Assuming transition inventories have been made zero, the first transition with $x_i \geq t_1$ must be to q to avoid shortage. Transition to greater than q would unnecessarily increase all subsequent $W(t)$. Thus any policy with more than one transition for $t \geq t_1$ can be replaced by one with at most one without increasing peak net expense.

1.13. *At Least One Transition For* $t \geq t_1$. Suppose the last transition is at $x_i < t_1$. A policy which is at least as good will be found in which the last transition is $\geq t_1$.

The last transition must be to at least q to avoid permanent shortage. Transition to a production rate greater than q affects $W(t)$ only in the $I(t)$ term which is then larger than necessary for all $t > x_i$. Consequently, transition to exactly q is at least as good as transition to any rate greater than q.

Instead of transition to q, however, let the transition be to $m_i = q - \epsilon$. This can only reduce $I(t)$ and therefore $W(t)$ for $t > x_i$, at least up to the next transition which now will be necessary. Schedule the next transition when the inventory runs out; namely, at

$$\begin{aligned} x_{i+1} &= t_1 + \frac{I(t_1)}{\epsilon} \\ &= t_1 + \frac{\frac{h}{2}(t_1 - x_i)^2 - \epsilon(t_1 - x_i)}{\epsilon} \\ &= x_i + \frac{\frac{h}{2}(t_1 - x_i)^2}{\epsilon} \end{aligned}$$

This will give $x_{i+1} > t_1$ provided ϵ is chosen small enough, which can always be done.

Now for $t > t_1$, $W_D(t)$ is decreasing linearly, and $I(t)$ is nonincreasing so that, except for transitions, $W(t)$ is decreasing at least linearly. Thus, by picking ϵ small enough, the final transition to q will not produce a net expense peak greater than the highest prior net expense peak. Henceforth, then, assume that any policy with final transition before t_1 is modified to a policy with final transition at $x_f \geq t_1$. Thus, there are at least two transitions in policies so modified.

1.14. *All Peaks Are B Peaks and Are Equal Except For a Final A Peak Which May Be Lower.* This condition is established as follows:

1.141. Expressions for net expenses and net expense peaks are developed for policies satisfying 1.11, 1.12, and 1.13.

1.142. For policies satisfying 1.11, 1.12, and 1.13 some possible forms of the peaks are shown not to apply for certain t.

1.143. Any policy satisfying 1.11, 1.12, and 1.13 can be modified to satisfy 1.14 without increasing operating capital requirements.

1.141. *Expressions for net expenses and peaks.* Using all symbols as previously defined, the expression for net expense is

$$W(t) = W_D(t) + bI(t) + Tn(t) \tag{4.1}$$

where

$$W_D(t) = \begin{cases} at - \frac{1}{2}(p - b)ht^2 & t \leq t_1 \\ at - \frac{1}{2}(p - b)ht_1^2 - (p - b)ht_1(t - t_1) & t \geq t_1 \end{cases}$$

$$n(t) = i, \qquad x_i < t < x_{i+1} \quad \text{or} \quad x_i^+ \leq t \leq x_{i+1}^-$$

The $+$ and $-$ superscripts are necessary only when using x_i in $W(t)$

and $n(t)$ since $W(x_i)$ and $n(x_i)$ are ambiguous but $W_D(x_i)$ and $I(x_i)$ are not.

$$I(t) = \begin{cases} m_i(t - x_i) - \dfrac{h}{2}(t^2 - x_i^2), & x_i \leq t \leq x_{i+1} \quad \text{and} \quad t \leq t_1 \\ m_i(t - x_i) - \dfrac{h}{2}(t_1^2 - x_i^2) & \\ \qquad\qquad - ht_1(t - t_1), & x_i \leq t \leq x_{i+1} \quad \text{and} \quad t \geq t_1 \\ 0, \quad x_f \leq t & \end{cases}$$

where

$$(4.2) \qquad m_i = \begin{cases} \dfrac{h}{2}(x_i + x_{i+1}), & x_{i+1} \leq t_1 \\ ht_1 - \dfrac{\dfrac{h}{2}(t_1 - x_i)^2}{(x_{i+1} - x_i)}, & x_i < t_1 \leq x_{i+1} \\ q, \quad i = f & \end{cases}$$

The expressions for $I(t)$ and m_i include the requirement of zero transition inventories. Note that, by the definition of m_i, a sequence of transition times $x_i, x_2, \ldots, x_f$ with $x_1 = 0$, $x_{f-1} < t_1$, $x_f \geq t_1$ defines a complete production policy satisfying 1.11, 1.12, and 1.13.

Define the time period $x_i^+ \leq t \leq x_{i+1}^-$, to be the ith interval and denote the net expense peak in this interval by W_i^*. Since $W(t)$ is quadratic in this interval, there can be only one maximum in the interval and it must have one of the following forms.

A. $$W_{iA}^* = W(x_i^+) \quad \text{where} \quad \left.\frac{dW}{dt}\right|_{x_i^+} < 0$$

B. $$W_{iB}^* = W(t'_i) \quad \text{where} \quad \left.\frac{dW}{dt}\right|_{t_i} = 0$$

C. $$W_{iC}^* = W(x_{i+1}^-) \quad \text{where} \quad \left.\frac{dW}{dt}\right|_{x_{i+1}^-} > 0$$

This classification holds for $i = f$, the final peak, if $x_{f+1} = \infty$ by convention.

Now, from (4.1),

$$(4.3)\quad \frac{dW}{dt} = \begin{cases} a - pht + bm_i, & x_i < t < x_{i+1} \text{ and } t \leq t_1 \\ a - (p - b)q - \dfrac{bh(t_1 - x_i)^2}{2(x_{i+1} - x_i)^2} < 0, & t_1 \leq t < x_{i+1} \\ a - (p - b)q < 0, & x_f < t \end{cases}$$

Thus peaks occurring at $t \geq t_1$ must be A peaks. Since A peaks occur only at transitions and there is exactly one transition $\geq t_1$, there is exactly one peak $\geq t_1$; namely, W_f^*, and it is an A peak.

Now, formally define

$$t'_i = \frac{1}{ph}(a + bm_i)$$

irrespective of the position of t'_i. Then, by (4.3),

$$\frac{dW}{dt} = ph(t'_i - t), \qquad x_i < t < x_{i+1} \text{ and } t \leq t_1$$

This holds irrespective of the position of the x_{i+1} (implicit in t'_i because of m_i).

Hence, for any peak occurring at $t \leq t_1$, it is an A peak if

$$t'_i < x_i$$

a B peak if

$$x_i \leq t'_i \leq x_{i+1}$$

a C peak if

$$x_{i+1} < t_i$$

Given x_i and x_{i+1}, m_i can be determined. Then t'_i can be determined and, therefore, the type of peak that applies is found. Finally, the exact height of the peak is given by

$$(4.41)\quad W_{iA}^* = W(x_i^+)$$

$$= \begin{cases} ax_i - \frac{1}{2}(p - b)hx_i^2 + iT, & i < f \\ ax_f - \frac{1}{2}(p - b)ht_1^2 - (p - b)ht_1(x_f - t_1) + fT, & i = f \end{cases}$$

$$(4.42)\quad \begin{aligned} W_{iB}^* &= W(t'_i) \\ &= \tfrac{1}{2}pht'^2_i + \tfrac{1}{2}bhx_i^2 - bm_ix_i + iT \end{aligned}$$

$$(4.43)\quad \begin{aligned} W_{iC}^* &= W(x_{i+1}^-) \\ &= ax_{i+1} - \tfrac{1}{2}(p - b)hx_{i+1}^2 + iT \end{aligned}$$

1.142. *Some peak forms not applicable for certain t.* It will now be shown that

(1) $W(t)$ has no A peaks for $0 < t \leq t'$ (where $t' = [a/h(p - b)]$ as previously defined).

(2) $W(t)$ has no C peaks for $t' \leq t$.

(3) W_j^* is a B peak, where the interval $x_j < t' \leq x_{j+1}$ is denoted by j.

(1) If any W_i^* is an A peak, it must occur at x_i^+ and must have $t'_i < x_i$. Thus it suffices to show that $t'_i < x_i$ implies $x_i > t'$. Now

$$t'_i = \frac{a + bm_i}{ph} < x_i$$

but $m_i > hx_i$ whether or not $x_{i+1} > t_1$ so that

$$\frac{a + bhx_i}{ph} < x_i$$

or

$$x_i > \frac{a}{h(p - b)} = t'$$

(2) If any W_i^* is a C peak, it must occur at x_{i+1} and must have $t'_i > x_{i+1}$. Thus, it suffices to show that $t'_i > x_{i+1}$ implies $x_{i+1} < t'$.

$$t'_i = \frac{a + bm_i}{ph} > x_{i+1}$$

but $m_i < hx_{i+1}$ so

$$\frac{a + bhx_{i+1}}{ph} > x_{i+1}$$

or

$$x_{i+1} < t'$$

(3) For W_j^*, $x_j < t' \leq x_{j+1}$ so, by (1) and (2), the peak can be neither A nor C and, therefore, must be of B form. Since there are at least the transitions $x_1 = 0$ and $x_f \geq t_1$, there is at least one B peak.

1.143. *Equal B peaks through* W_{f-1}^* *and* $W_{fA}^* \leq W_1^*$. Consider two successive peaks W_i^* and W_{i+1}^* involving transition times x_i, x_{i+1}, x_{i+2}. (If $i + 1 = f$, think of $x_{i+2} = \infty$.) Policies will be modified by holding x_i and x_{i+2} constant and varying x_{i+1} so as to change these two peaks. As long as x_i and x_{i+2} are constant (4.3) shows that $W(t)$ outside these two intervals is unchanged except that, if x_{i+1} is made to coincide with x_i or x_{i+2}, a transition is eliminated and $W(t)$ is reduced for $t > x_{i+2}$.

For $i + 1 \leq j$ the two peaks can only be C or B. In the case of W_i^*, if it is a B peak,

$$\frac{dW_{iB}^*}{dx_{i+1}} = \frac{bh}{2}(t'_i - x_i) \geq 0$$

and if it is a C peak,

$$\frac{dW_{iC}^*}{dx_{i+1}} = h(p - b)(t' - x_{i+1}) > 0$$

since $x_{i+1} < t' < t_1$.

In the case of W_{i+1}^*, if it is a B peak,

$$\frac{dW_{i+1,B}^*}{dx_{i+1}} = b(t'_{i+1} - x_{i+1})\frac{dm_{i+1}}{dx_{i+1}} - b(m_{i+1} - hx_{i+1})$$

$$= \begin{cases} -\dfrac{bh}{2}(x_{i+2} - t'_{i+1}) & \text{if} \quad x_{i+2} \leq t_1 \\ -\dfrac{bh(t_1 - x_{i+1})(x_{i+2} - x_{i+1} + x_{i+2} - t_1)(x_{i+2} - t'_{i+1})}{2(x_{i+2} - x_{i+1})^2}, & \text{if} \quad x_{i+2} > t_1 \end{cases}$$

or

$$\frac{dW_{i+1,B}^*}{dx_{i+1}} \leq 0$$

since

$$x_{i+1} \leq t'_{i+1} \leq x_{i+2} \quad \text{if} \quad x_{i+2} \leq t_1$$
$$x_{i+1} \leq t'_{i+1} \leq t_1 < x_{i+2} \quad \text{if} \quad x_{i+2} > t_1$$

If it is a C peak,

$$\frac{dW_{i+1,C}^*}{dx_{i+1}} = 0$$

Now W_i^* and W_{i+1}^* are continuous in x_{i+1} for $x_i < x_{i+1} < x_{i+2}$. (Equality at either end would, however, eliminate a transition and a peak.) In other words, there is no discontinuity in a peak if there is a change from B to C or vice versa from varying x_{i+1}. Hence, for $i < j$,

$$\frac{dW_i^*}{dx_{i+1}} \geq 0 \quad \text{and} \quad \frac{dW_{i+1}^*}{dx_{i+1}} \leq 0$$

For $i \geq j$, W_i^* and W_{i+1}^* can be only A or B peaks. If W_i^* is a B peak,

$$\frac{dW_{iB}^*}{dx_{i+1}} = b(t'_i - x_i)\frac{dm_i}{dx_{i+1}} \geq 0$$

and if W_i^* is an A peak,

$$\frac{dW_{iA}^*}{dx_{i+1}} = 0$$

In the case of W_{i+1}^*, if it is a B peak,

$$\frac{dW_{i+1,\,B}^*}{dx_{i+1}} \leq 0$$

as established earlier; and if W_{i+1}^* is an A peak,

$$\frac{dW_{i+1,\,A}^*}{dx_{i+1}} = \begin{cases} h(p - b)(t' - x_{i+1}) \leq 0, & x_{i+1} \leq t_1 \\ h(p - b)(t' - t_1) \leq 0, & x_{i+1} \geq t_1, \end{cases} \qquad (i + 1 = f)$$

Once again the peaks are continuous in x_{i+1} so that

$$\frac{dW_i^*}{dx_{i+1}} \geq 0 \quad \text{and} \quad \frac{dW_{i+1}^*}{dx_{i+1}} \leq 0$$

for $i \geq j$ and thus for all pairs of peaks $(i,\ i + 1)$ no matter what i.

Now, take a pair of peaks. Move x_{i+1} in the direction of the larger peak until the peaks are equal or a transition is eliminated. Starting with $i = 1$, do this for successive pairs, repeating the sequence as necessary until all peaks remaining are equal. In case a move brings $x_f < t_1$, create a new $x_f \geq t_1$ according to the discussion in 1.13.

Any peak with $i < j$ must be a B peak. Suppose this is not true. Then there is a C peak, W_{iC}^*. However,

$$W_{iC}^* = W(x_{i+1}^-) < W(x_{i+1}^-) + T = W(x_{i+1}^+) \leq W_{i+1}^*$$

The strict inequality contradicts the equality previously established. Hence, since $i = j$ is a B peak, all peaks through j are B peaks.

Next move along through i to the first A peak. If it is W_f^*, the original assertion is satisfied. If not, let i refer to the first A peak.

$$W_{iA}^* = W_{i+1}^* = W_j^*$$

Now, since

$$\frac{dW_{iA}^*}{dx_{i+1}} = 0 \quad \text{and} \quad \frac{dW_{i+1}^*}{dx_{i+1}} \leq 0$$

x_{i+1} can be increased without increasing any peaks until a transition is eliminated or until W_i^* just becomes a B peak with $W_{iB}^* = W_j^*$. (Increasing x_{i+1} increases t'_i.) This leaves

$$W_{i+1}^* \leq W_j^*$$

If $i + 1 = f$, the process is finished. If not, consider

$$W_{i+1}^* \leq W_{i+2}^* = W_j^*$$

In the light of the derivatives with respect to x_{i+2}, x_{i+2} can be increased until a transition is eliminated or $W_{i+1}^* = W_j^*$. This may leave an $i + 2$ for which

$$W_{i+2}^* \leq W_j^*$$

but repeated application will set all peaks equal to W_j^* with the possible exception of W_f^* which may be lower.

Return now and look again for A peaks. If there are any except W_f^*, take the one with smallest i and repeat the procedure above. Continuing, eventually all peaks will be B peaks equalling W_j^* except for W_f^* which will be an A peak and may be lower.

1.2 An Optimal Policy. A particular policy in the class obtained by modifying any policy according to the steps in 1.1 will be identified and shown to be optimal. The steps in this section are:

1.21. Necessary and sufficient conditions are given to have a policy which is in the class described in 1.1 and which has a peak capital requirement of K.

1.22. A procedure for finding an optimal policy is described.

1.21. *Necessary and Sufficient Conditions.* Consider only policies satisfying 1.11, 1.12, and 1.13; that is, policies in which there is no inventory at transitions and in which there is one and only one transition for $t \geq t_1$ and that a transition to q. Then any sequence $x_1, \ldots, x_f$ ($x_1 = 0$, $x_{f-1} < t_1$, $x_f \geq t_1$) of transition times defines a production policy with production rates given by (4.2) and having peaks in each interval as found in (4.4).

It will be shown that the conditions (Ia)

$$x_{i+1} = \begin{cases} \dfrac{2}{bh}\left[\left(p - \dfrac{b}{2}\right)hx_i - a + \sqrt{N_i}\right], & i + 1 < f \qquad (4.51) \\[2ex] x_i + \dfrac{\dfrac{bh}{2}(t_1 - x_i)^2}{bht_1 + a - phx_i - \sqrt{N_i}}, & i + 1 = f \qquad (4.52) \end{cases}$$

where

$$N_i = 2ph[K - iT - W_D(x_i)]$$

$$x_1 = 0$$

$$f = \text{first value of } i + 1 \text{ such that } x_{i+1} \geq t_1 \text{ in (4.51)}$$

There is a K such that (Ib)

$$N_i \geq 0, \qquad i \leq f \tag{4.61}$$

$$+\sqrt{N_i} \geq \frac{2ph(p - b)}{2p - b}(t' - x_i), \qquad i < f - 1 \tag{4.62}$$

and

$$+\sqrt{N_i} \leq ph(t_1 - x_i), \qquad i = f - 1 \tag{4.63}$$

are necessary and sufficient for having a policy $(x_1, \ldots, x_f)$ with a sequence of peaks which satisfy (II)

$$W_i{}^* = W_{iB}{}^* = K, \qquad i < f \tag{4.71}$$

and

$$W_i{}^* = W_{iA}{}^* \leq K, \qquad i = f \tag{4.72}$$

If there is a policy for which (II) holds,

$$W_i{}^* = W_{iB}{}^* = K, \qquad i < f$$

so that

$$K = \tfrac{1}{2}pht'^2_i + \tfrac{1}{2}bhx_i^2 - bm_ix_i + iT \tag{4.8}$$

and

$$x_i \leq t'_i \leq x_{i+1} \tag{4.9}$$

where

$$t'_i = \frac{a + bm_i}{ph}$$

and m_i is given in terms of x_i and x_{i+1} by (4.2). Next, solve for $m_i(x_i)$ then $x_{i+1}(m_i)$ to get (Ia). From (4.8),

$$K - iT - W_D(x_i) = \frac{(a + bm_i)^2}{2ph} + \frac{bh}{2}x_i^2 - bm_ix_i - W_D(x_i)$$

and, since $x_i < t_1$ for $i < f$,

$$\frac{N_i}{2ph} = \frac{(a + bm_i)^2}{2ph} - bm_ix_i - ax_i + \frac{1}{2}phx_i^2$$

Solving the quadratic in $a + bm_i$,

$$a + bm_i = phx_i \pm \sqrt{N_i}$$

$$m_i = \frac{-a + phx_i \pm \sqrt{N_i}}{b}$$

hence

$$t'_i = x_i \pm \frac{1}{ph}\sqrt{N_i}$$

By (4.9) only the positive square root is admissible so that

$$m_i = \frac{-a + phx_i + \sqrt{N_i}}{b}$$

Using (4.2) express x_{i+1} in terms of x_i for $i = 1, \ldots, f - 1$. The results are (4.51) and (4.52).

Next, K must be such that $N_i \geq 0$ for $i < f$ in order to have real x_{i+1}. For $i = f$, in order to have

$$W_f^* = W_{iA}^* \leq K$$

$$W_{iA}^* = fT + W_D(x_f) \leq K$$

or

$$N_f = 2ph[K - fT - W_D(x_f)] \geq 0$$

thereby establishing (4.61).

Now, in order to have B peaks,

$$t'_i \leq x_{i+1}, \qquad i + 1 < f$$

$$t'_i \leq t_1, \qquad i + 1 = f$$

In the first case, in order to have

$$(x_{i+1} - t'_i) \geq 0$$

$$\left(p - \frac{b}{2}\right)hx_i - a + \sqrt{N_i} - \frac{bh}{2}t'_i \geq 0$$

or K must be such that

$$+\sqrt{N_i} \geq \frac{2ph(p-b)}{2p-b}(t' - x_i), \qquad i < f - 1$$

In the second case, to have

$$t_1 - t'_i \geq 0 \quad \text{when} \quad i + 1 = f$$

$$t_1 - x_i - \frac{1}{ph}\sqrt{N_i} \geq 0$$

or

$$+\sqrt{N_i} \leq ph(t_1 - x_i)$$

so that (4.62) and (4.63) are established.

If (I) holds, it will first be shown that (I) leads to a policy of the class being considered; that is, to an increasing sequence of x_i terminating with $x_{f-1} < t_1$ and $x_f \geq t_1$.

Using (4.51)

$$(x_{i+1} - x_i) = \frac{2(p-b)}{b}\left[x_i - t' + \frac{1}{h(p-b)}\sqrt{N_i}\right]$$

If $x_i > t'$, this is positive by (4.61) and, in fact, the length of interval is greater than an increasing quantity. If $x_i < t'$, (4.62) gives

$$(x_{i+1} - x_i) \geq \frac{2(p-b)}{(2p-b)}(t' - x_i) \geq 0$$

It appears that x_i might conceivably converge to t' but this is not possible because N_i would have to equal 0 or be arbitrarily close to 0. However, K being fixed,

$$N_{i+1} = N_i - 2phT - 2ph[W_D(x_{i+1}) - W_D(x_i)]$$

The contents of the bracket is positive for $x_{i+1} < t'$ so that N_{i+1} would be driven negative in violation of (4.61). Thus, if (Ib) is satisfied, x_{i+1} will eventually jump t'. Since the steps thereafter are greater than an increasing quantity, eventually (4.51) will formally give

$$x_{i+1} \geq t_1$$

In this case, a little algebra shows that (4.52) will give

$$x_{i+1} \geq t_1$$

if (4.51) does and x_f has been found.

Next it will be shown that there are B peaks for $i < f$.

$$t'_i = \frac{a + bm_i}{ph}$$

Substitution of (4.51) into the expressions for m_i in (4.2) and this into the equation above gives

$$t'_i = x_i + \frac{\sqrt{N_i}}{ph}, \qquad i < f$$

so that $x_i \leq t'_1$ by (4.61). Now, using (4.51)

$$\begin{aligned}(x_{i+1} - t'_i) &= \frac{2}{b}(p - b)\left[x_i - t' + \frac{(2p - b)}{2ph(p - b)}\sqrt{N_i}\right] \\ &\geq 0 \quad \text{if} \quad i + 1 < f\end{aligned}$$

by (4.62).

Finally, for $i + 1 = f$,

$$(t_1 - t'_i) = t_1 - x_i - \frac{\sqrt{N_i}}{ph} \geq 0$$

by (4.63). Thus $x_i \leq t'_1 \leq x_{i+1}$ for $i + 1 < f$ and $x_i \leq t'_i \leq t_1$ for $i + 1 = f$ so there are B peaks for $i < f$. These B peaks are all equal because

$$\begin{aligned}W_{iB}^* &= \frac{ph}{2}t'^2_i + \frac{bh}{2}x_i^2 - bm_ix_i + iT \\ &= K, \qquad i < f\end{aligned}$$

after the appropriate substitutions.

Finally W_f^* is an A peak because $x_f \geq t_1$ as shown in 1.1. In addition,

$$W_{fA}^* = W_D(x_f) + fT$$

but

$$N_f = 2ph[K - fT - W_D(x_f)] \geq 0$$

by (4.61) so

$$W_{fA}^* \leq K$$

1.22. *A Procedure for Finding an Optimal Policy.* First of all, there exists a policy of the type given in II above. For, take any policy (for example, $x_1 = 0$, $x_2 = t_1$ defines a policy with some calculable peak capital requirement which is finite) and modify it by the procedures of 1.1. The result is a policy of the type in II with some value of K.

Now reduce K until the smallest value is reached such that, when the policy of (Ia) is constructed, none of the restrictions of (Ib) are violated. There must be such a smallest value since no $K \leq 0$ will permit satisfying the restrictions. Call the smallest K, W^* and the resulting policy P.

W^* is the minimum peak capital requirement and P is an optimal policy. For, suppose there is a better policy P' with peak capital requirement $W^{*\prime} < W^*$. By 1.1 this can be transformed into a new policy of the class described by II with a peak requirement $K \leq W^{*\prime} < W^*$. However, W^* is the smallest K for policies in this class, which contradicts the assumption of a better policy. Thus, P is optimal.

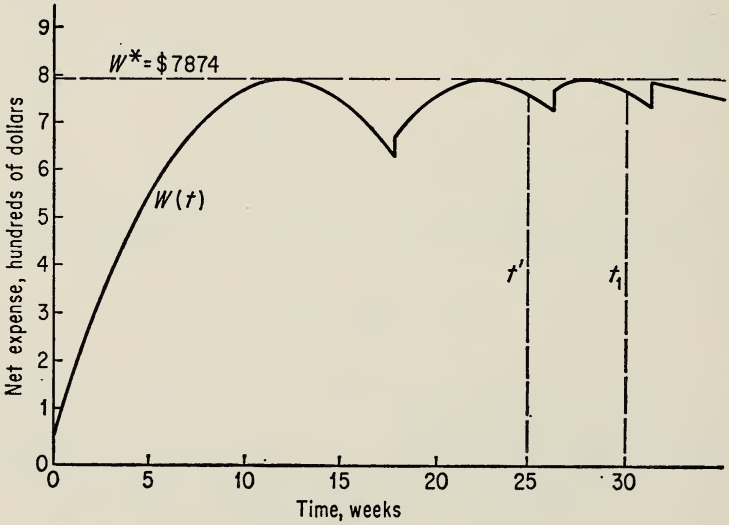

Fig. 4.1. Net expense as a function of time

W^* and P may be calculated in practice from (Ia) by starting with a high K and reducing it to the lowest value which will not violate (Ib). This is not difficult graphically and would be even easier on a computer.

As a numerical example of the foregoing, let

$T = \$500$

$a = \$500$ per week

$b = \$80$ per unit of product

$h = 1$ unit of product per week squared

$p = \$100$ per unit of product

$q = 30$ units per week

$t' = 25$ weeks

By trial and error, W^* is found to be \$7874 for this example and the policy is:

i	x_i	m_i
1	0	8.93
2	17.86	22.03
3	26.20	28.57
4	31.26	30.00

Net expense $W(t)$ as a function of time is plotted in Fig. 4.1 along with W^*, t', and t_1.

4.3. Maximizing Profit for the Initiation Period

On the assumption that ample working capital is available to meet initial operating capital requirement, the criterion of maximizing initiation period profit becomes applicable. Since gross revenue will not be affected by any admissible production policy, maximizing profits amounts to minimizing costs. With a linear production cost function, a given output costs the same regardless of the rates at which it was produced. Consequently, the only costs which can be affected by the production policy (assuming all produced output must be sold) are inventory and transition costs. Call the total of these costs to time t, $C(t)$.

$$C(t) = s \int_0^t I(u)du + Tn(t)$$

where s is inventory cost per unit of product per unit time, $I(t)$ is inventory as a function of time, T is the cost of a transition in output rate, $n(t)$ is the number of transitions up to time t. Now $C(x_f)$, where x_f is the time of the final transition to q, is to be minimized.

First, note that there is an optimal policy in which $I(x_i) = 0$ for all transition points. The argument of Section 1.11 may be used again.

That argument made use of the fact that reducing I at t can only reduce W at t. This fact needs only to be replaced with the observation that reducing I at $t \leq x_{i+2}$ can only reduce $C(x_{i+2})$.

Consider any three consecutive transition points, x_i, x_{i+1}, and x_{i+2} with $x_{i+2} \leq t_1$. Inventory for the first rate is

$$I(t) = I(x_i) + m_i(t - x_i) - \frac{h}{2}(t^2 - x_i^2), \qquad x_i \leq t < x_{i+1} \leq t_1$$

where m_i is the production rate. With zero transition inventories,

$$m_i = \frac{h}{2}(x_i + x_{i+1})$$

and

$$I(t) = \frac{h}{2}(x_i + x_{i+1})(t - x_i) - \frac{h}{2}(t^2 - x_i^2)$$

Inventory cost during the interval is

$$K_i(t) = s \int_{x_i}^{t} I(u)du$$

from which it follows that

$$K_i(x_{i+1}) = \frac{sh}{12}(x_{i+1} - x_i)^3$$

$$K_{i+1}(x_{i+2}) = \frac{sh}{12}(x_{i+2} - x_{i+1})^3$$

Hence,

$$\frac{dK_i(x_{i+1})}{dx_{i+1}} = \frac{sh}{4}(x_{i+1} - x_i)^2 > 0$$

$$\frac{dK_{i+1}(x_{i+2})}{dx_{i+1}} = -\frac{sh}{4}(x_{i+2} - x_{i+1})^2 < 0$$

Inventory and transition costs up to x_{i+2} are

$$C(x_{i+2}) = C(x_i) + K_i(x_{i+1}) + K_{i+1}(x_{i+2}) + 2T$$

Therefore,

$$\frac{dC(x_{i+2})}{dx_{i+1}} = \frac{sh}{4}(x_{i+2} - x_i)(2x_{i+1} - x_i - x_{i+2})$$

and

$$x_{i+1} = \frac{x_i + x_{i+2}}{2} \quad \text{minimizes} \quad C(x_{i+2})$$

that is,

$$\min\,[C(x_{i+2})] = C(x_i) + 2T + \frac{sh}{6}\left(\frac{x_{i+2} - x_i}{2}\right)^3$$

Given a transition point $x_i < t_1$ and given that there are to be k transitions up to x_i (including $x_i = 0$ and $x_i = x_k$), the form for $\min\,[C(x_{i+2})]$ makes it evident that the lowest cost up to x_k is obtained by dividing the time into $k - 1$ equal intervals.

Now consider the last two transition points, x_{f-1} and x_f (where $x_f \geq t_1$), together with the production rate between them, m_f. In order for transition inventories to be zero

$$m_f(x_f - x_{f-1}) = \frac{h}{2}(t_1^2 - x_{f-1}^2) + q(x_f - t_1)$$

or

$$m_f = \frac{h(2t_1x_f - x_{f-1}^2 - t_1^2)}{2(x_f - x_{f-1})}$$

Now

$$\begin{aligned} K_f(x_f) &= s\int_{x_{f-1}}^{t_1} I(u)du + s\int_{t_1}^{x_f} I(u)du \\ &= s\,\frac{(t_1 - x_{f-1})^2}{6}\,[3m - h(t_1 + 2x_{f-1})] + \frac{sh(t_1 - x_{f-1})^2\,(x_f - t_1)^2}{4(x_f - x_{f-1})} \\ &= \frac{sh(t_1 - x_{f-1})^2}{12(x_f - x_{f-1})}\,(x_f^2 + 3x_f^2 + 2x_{f-1}t_1 - 2x_ft_1 - 4x_fx_{f-1}) \end{aligned}$$

It follows that

$$\frac{dK_f(x_f)}{dx_f} = \frac{sh}{4}(t_1 - x_{f-1})^2 > 0$$

Hence, lowering x_f to t_1 will minimize inventory costs for the last rate. However, if the last rate ends at t_1, the previous analysis applies. Therefore, for minimum initiation period costs and maximum profits, t_1 should be divided into n equal time intervals (where $n + 1$ is the number of transitions) and production rates should be set to make all transition inventories zero.

There remains the problem of finding the optimal number of production rates, n_0. Total relevant costs over the initiation period are

$$\begin{aligned} C(t_1) &= (n+1)T + n\left[\frac{sh}{12}\left(\frac{q}{nh}\right)^3\right] \\ &= (n+1)T + \frac{sq^3}{12n^2h^2} \end{aligned}$$

The optimal number of intermediate production rates n_0 must satisfy

$$C(n_0 - 1) > C(n_0) < C(n_0 + 1)$$

or

$$\frac{2n_0 - 1}{n_0^2(n_0 - 1)^2} > \frac{12h^2T}{sq^3} > \frac{2n_0 + 1}{n_0^2(n_0 + 1)^2}$$

To expedite the process of finding n_0 treat n as a continuous variable. Then

$$\frac{dC(t_1)}{dn} = T - \frac{sq^3}{6h^2n^3}$$

from which

$$n'_0 = \sqrt[3]{\frac{sq^3}{6h^2T}}$$

if n'_0 is integral. If not, the nearest integer above and below n'_0 can be used in $C(t_1)$ to find n_0.

CHAPTER 5

Decisions With Probabilistic Demand

5.1. Summary

In order to reduce complexity, previous chapters have assumed that demand as a function of time is known. The present chapter relaxes the assumption of known demand. Hence, it is concerned with the most important decision of all: whether or not to go into business in the face of uncertain demand. Two problems which affect this decision are treated.

First, even if no variables affecting the amount of starting capital are subject to control by the prospective firm's management, an estimate of the probability of success is needed. Given certain data, a method for calculating this probability is developed for three situations: availability of unlimited equity capital, availability of a specified amount of equity capital and no loan amount, availability of a specified amount of equity capital together with a loan of given amount, and a specified pay-back deadline.

When some freedom of action is left to the management, decisions should maximize the probability of success. With respect to initial financing, three possible control variables present themselves. The amount of money available after set-up to cover the initial operating loss certainly affects the probability of success. However, the variety of sources for equity capital and the variety of types and degrees of control concessions demanded by lenders have prevented development of a reasonably realistic general relationship between amount of capital which can be had and the cost of obtaining it. Nonetheless, knowing how the probability of success varies with the amount of funds on hand when set-up is completed should provide a highly valuable aid to comparison of alternative financing opportunities.

Another possible decision variable is the amount of money to be borrowed to cover remaining unpaid set-up costs as well as initial operating losses. The maximum, however, is typically decided by a lending agency on the basis of an appraisal of assets, net worth, and managerial skill. Usually even the maximum is conservative and the entire amount is borrowed.

Finally, the time period at the end of which a loan must be repaid can be an effective decision variable. Within safe maximum limits, lending agencies prefer to spread their set-up cost of loans over a long interest-earning period. Borrowers prefer to reduce the loan period in order that as little discount as possible will be subtracted from the amount of the loan. The second problem considered in this chapter, therefore, is that of setting the loan period to maximize the probability of success.

The model used throughout the chapter is for firms characterized by time, speed, and unit segmentation (production costs increase linearly with output rate). In addition, continuous divisibility of output rate without transition cost is assumed possible. Fixed costs are taken to be a feature of the entire plant rather than of the number of work centers operating. Finally, the demand and revenue functions used are those described for a cash-on-delivery business in the first part of Chapter 2.

5.2. Joint Density of Growth Rate and Ultimate Demand

Although demand is assumed to grow linearly at a rate of h up to an ultimate, stable rate of q, the numerical values of these parameters are not known in advance. It is anticipated that such factors as product; number, composition, and income of potential customers; size of firm in terms of productive facilities; sales volume of direct competitors; and price range on product could be used to isolate a group of firms similar to the new firm of interest. Given this group of similar firms together with the case histories on their demand experience, paired values of ultimate demand rate achieved q and demand growth rate experienced h could be determined for each firm of the class. By their relative frequency pattern over the qh-plane, the collection of such cases comprises an empirical joint density function $f(q, h)$, which expresses the likelihood of meeting or exceeding a given ultimate demand rate q_1 by linear growth at a rate at least as large as a specified value h_1.

It must be pointed out that neither the joint density functions described nor all the data necessary to construct them are now available in public sources for the various industries. It would seem, however, that reports by businesses to the Bureau of Internal Revenue, Federal Security Agency, and Bureau of the Census could be combined with such publications as the *Survey of Buying Power* published by Sales Management Magazine to provide the necessary data for many types of firms. If these sources proved unsatisfactory, polls by trade associations on a periodic basis offer another possibility.

The joint density function described is, of course, three-dimensional, but a two-dimensional scatter plot will be used to represent the function. In Fig. 5.1, each dot locates a pair of q, h values which together represent the demand case history of one firm in the relevant class.

5.3. Probability of Success — Analytic Calculation

Even if the subject new firm were possessed of unlimited funds, success is still not assured. Ultimate demand q must be sufficient to provide at least enough revenue to cover costs. That is,

$$q > \frac{a}{p - b} \tag{5.1}$$

in which the fixed cost rate is a, p is price, and b is variable cost. Therefore,

$$P(\text{success/unlimited funds}) = \int_{a/p-b}^{\infty} dq \int_{0}^{\infty} f(q, h)dh$$

When production costs increase linearly with output and when there are no transition costs, it will be recalled that the proper production policy for minimizing the initial operating capital requirement is to produce to demand. The initial operating capital requirement in this case is

$$W^* = \frac{a^2}{2h(p - b)}$$

If M is defined as the amount available to cover initial operating losses after set-up has been paid for and loans have been made, then, for the firm to succeed,

$$M \geq W^*$$

or

$$h \geq \frac{a^2}{2(p - b)M} \tag{5.2}$$

Hence, if there is no pay-back deadline as would be the case if, for example, M were all equity capital,

$$P(\text{success}/M, \text{ no deadline}) = \int_{a/p-b}^{\infty} dq \int_{a^2/2(p-b)M}^{\infty} f(q, h)dh$$

As the last case in this section, assume that at least a part of M is the discounted proceeds of a loan of amount L which must be repaid at the end of t_L. Then, to achieve success,

$$L \leq \begin{cases} M - W^*, & t_L < \dfrac{a}{h(p - b)} \quad (5.31) \\ M - at_L + \dfrac{h(p - b)t_L^2}{2}, & \dfrac{a}{h(p - b)} < t_L < \dfrac{q}{h} \quad (5.32) \\ M - \dfrac{q^2(p - b)}{2h} + [q(p - b) - a]t_L, & \dfrac{q}{h} < t_L \quad (5.33) \end{cases}$$

Equation (5.31) states that, if the loan must be repaid before revenue rate has become as great as cost rate, M must be great enough to cover both the loan at t_L and the net operating loss W^* the entirety of which will not be spent until

$$t' = \frac{a}{h(p - b)}$$

Equations (5.32) and (5.33) state that, at t_L, the cumulative net revenue that the firm has earned since commencing operations together with the unspent portion of M must be at least as large as the amount borrowed.

If (5.31) applies, then the loan does not increase the new firm's ability to cover W^*. If the firm does succeed under these conditions, it must have had enough money without the loan proceeds at the commencement of operations to cover W^* and to pay the discount on the loan. Hence, making the loan would create an unnecessary expense in the amount of the discount paid. The remaining discussion will exclude this case; that is, it will be assumed that $L > M - W^*$. However, since L and M are constants and since W^* is a random variable which approaches zero as h grows large without limit, it is necessary to assume $L > M$ (the loan exceeds available starting capital at $t = 0$) in order to exclude (5.31) for all permissible values of h. In effect, then, excluding (5.31) means that success is impossible unless

$$h > \frac{a}{(p-b)t_L} \tag{5.4}$$

Given L, M, p, b, a, and t_L, the probability of success can be found by integrating $f(q, h)$ over those pairs of q and h values which simultaneously satisfy (5.1), (5.2), (5.4), and an appropriate one of (5.32) and (5.33). However, if either (5.32) or (5.33) holds, (5.1) and (5.4) are redundant. Suppose (5.32) holds. Then, the conditions on t_L are

$$\frac{a}{h(p-b)} < t_L < \frac{q}{h}$$

which implies (5.1). Since $L > M$, (5.32) can be restated as a condition on h.

$$0 \leq L - M \leq \frac{h(p-b)t_L^2}{2} - at_L$$

or

$$0 \leq \frac{2(L-M) + 2at_L}{t_L^2(p-b)} \leq h$$

Now,

$$\frac{2a}{t_L(p-b)} + \frac{2(L-M)}{t_L^2(p-b)} > \frac{a}{t_L(p-b)}$$

Therefore, if $L - M > 0$ and if (5.32) holds, (5.4) is redundant.

Next, suppose (5.33) holds. Then, since $L > M$,

$$0 < L - M \leq (p-b)\left[t_L\left(q - \frac{a}{p-b}\right) - \frac{q^2}{2h}\right]$$

which implies (5.1). Furthermore, $q/h < t_L$ or $h > q/t_L$. Now,

$$h > \frac{q}{t_L} > \frac{a}{(p-b)t_L} \quad \text{if} \quad q > \frac{a}{p-b}$$

The latter was shown to apply if $L > M$. Therefore, (5.4) is redundant.

By extending the argument a little further, the smallest q for any possible h can be found. For fixed t_L, the smallest possible q is found from (5.33). Given that $L > M$, then

$$q \geq \frac{L - M}{t_L(p - b)} + \frac{a}{p - b} + \frac{q^2}{2ht_L}$$

and the smallest permissible value of q results when h grows large without limit. This minimum value is

$$q_L = \frac{L - M}{t_L(p - b)} + \frac{a}{p - b} \tag{5.5}$$

Equations (5.2), (5.32), and (5.33) together express conditions on h which must be met if the firm is to succeed. Put in terms of h, these equations state that

$$h \geq \frac{a^2}{2M(p - b)} \tag{5.61}$$

$$h \geq \frac{2[at_L + (L - M)]}{t_L{}^2(p - b)}, \qquad ht_L < q \tag{5.62}$$

$$h \geq \frac{q^2(p - b)}{2\{t_L[q(p - b) - a] - [L - M]\}}, \qquad ht_L > q \tag{5.63}$$

Now let

$$h' = \max \begin{cases} \dfrac{a^2}{2M(p - b)} \\[2ex] \dfrac{2[at_L + (L + M)]}{t_L{}^2(p - b)} \end{cases}$$

and let

$$h_L(q) = \begin{cases} h', & q > q_B \\[1ex] \dfrac{q^2(p - b)}{2\{t_L[q(p - b) - a] - [L - M]\}}, & q < q_B \end{cases}$$

where q_B is found from equating the two expressions for $h_L(q)$. It can readily be shown that, as q approaches q_L from above, h in (5.63) grows large without limit. Therefore the second expression for $h_L(q)$ will be greater than h' for $q < q_B$.

In order for the firm to succeed, given known parameters L, M, t_L, p, b, and a,

$$h \geq h_L(q) \quad \text{and} \quad q \geq q_L$$

Hence,

$$P(\text{success} \mid M, L, t_L) = \int_{q_L}^{\infty} \int_{h_L(q)}^{\infty} f(q, h)dh\, dq$$

5.4. Probability of Success — Graphical Calculation

The situation described above can be clarified by use of a probability analysis graph, Fig. 5.1. The density $f(q, h)$ can be thought of as being plotted up out of the page. However, as a practical device, the actual experience of the class of similar hypothetical firms is shown as a scatter plot in the q, h-plane. Once $h_L(q)$ has been constructed (as illustrated in the figure by the solid line) the percentage of firms lying above and to the right of this line is easily ascertained.

The calculation of the $h_L(q)$ line shown in Fig. 5.1 proceeds as follows. Assume that, at the completion of set-up, \$1000 is owed to a contractor and that the only available lending agency will lend \$5000 for exactly 50 weeks at a discount rate of 0.2 per cent per week. Further assume that for the firm in question a = \$500 per week, p is \$100 per unit of product and b is \$80 per unit of product. Then

$$M = Le^{-it_L} - \$1000 = \$3524$$

$$\frac{a^2}{2M(p - b)} = 1.77$$

$$\frac{2[at_L + (L - M)]}{t_L{}^2(p - b)} = 1.06$$

Therefore, $h_L(q) = h' = 1.77$ which applies for all $q > 32.41$. As q decreases, the second expression for $h_L(q)$ becomes applicable and this function approaches $q_L = 26.476$ as found from (5.5). These functions are illustrated in Fig. 5.1 from which it is evident that the percentage of firms above $h = 1.77$ and to the right of

$$h_L(q) = \frac{q^2}{100q - 2647.6}$$

represents the probability of success.

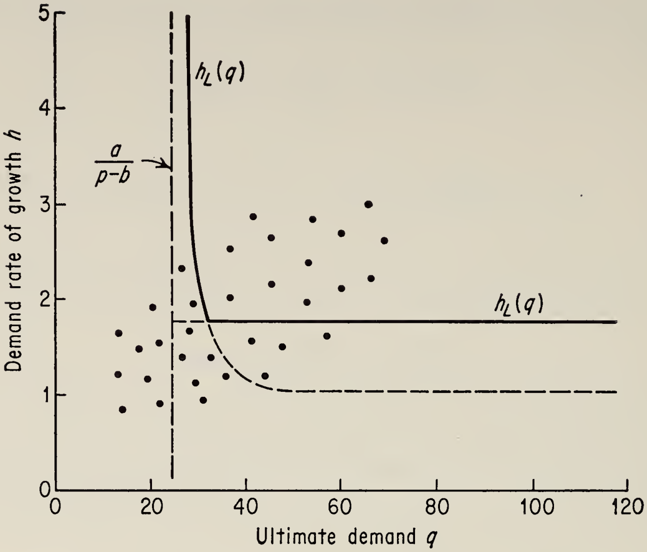

Fig. 5.1. Probability of success

The probability of success, given unlimited funds, is composed of the percentage of firms to the right of

$$q = \frac{a}{p - b} = 25$$

The probability of success given an M of \$3524 and no repayment deadline is the percentage of firms above $h = 1.77$ and to the right of $q = 25$.

5.5. Maximizing the Probability of Success

It is possible that a new firm, after the only available lending agency has considered its potential, is offered a loan with specified amount, time period, and interest rate. Under these conditions no flexibility remains and the firm must either take the loan or leave it. As has been illustrated, the probability of success would be calculated and this would contribute to the decision whether to go ahead or not. It is also possible that the firm would be offered a small number of alternative loan amounts with the time period and interest rate specified for each. Here, too, the probability of success would be calculated for each and the decision would be made on the basis of the highest such probability. There is also the possibility that some lending agency would be willing to increase the amount loaned if the length of the loan period were decreased or if a higher rate of interest were acceptable. Where a quantitative relationship between the variables can be specified, that is,

$$L = L(i, t_L)$$

then i and t_L become decision variables. The insertion of $L(i, t_L)$ into the expressions for q_L and $h_L(q)$ and thence into the expression for the probability of success makes the latter a function of the decision variables. Thus, conceptually at least, the probability of success can be maximized. For the most part it is likely to be difficult to specify a function of this type that would have general utility. Therefore, the discussion will be restricted to the following possibility: the businessman is offered a loan of specified amount at a specified rate of interest but he is given some latitude as to the time period permitted before the loan must be repaid.

If t_L is long, the new firm will have sufficient time to generate the money to repay the loan. However, since the cash yield of the loan is only

$$Le^{-it_L}$$

there will be less cash to cover W^* than if t_L were shorter. That an optimal t_L exists can be seen intuitively from noting that both $t_L = \infty$ and $t_L = 0$ reduce the amount available to help cover W^* to nothing. Inserting the yield expression above into an expression for M will make probability of success a function of t_L. Analytic maximization appears to be excessively unwieldy but a graphical solution where $f(q, h)$ is represented by a scatter plot and which is sufficiently accurate for practical purposes is relatively easy. Such a solution has been carried out in Fig. 5.2 except for integration or counting points. Thus the functions graphed in the figure can be superimposed on any $f(q, h)$. To describe Fig. 5.2, it will be assumed that

$L = \$5000$

$i = 0.2$ per cent per week

$a = \$500$ per week

$p = \$100$ per unit product

$b = \$80$ per unit product

and that

$$M = Le^{-it_L} - \$1000$$

as before.

Then Table 5.1 can be calculated from equations (5.61) and (5.62).

TABLE 5.1

t_L	M	h_1	h_2
20	\$3804	1.64	2.80
25	3756	1.66	2.20
30	3709	1.69	1.81
35	3662	1.71	1.54
40	3616	1.73	1.34
45	3570	1.75	1.18
50	3524	1.77	1.06

The values of h_1 in Table 5.1 are calculated from equation (5.61) and the values of h_2 from equation (5.62). The values of h_2 will hold for all

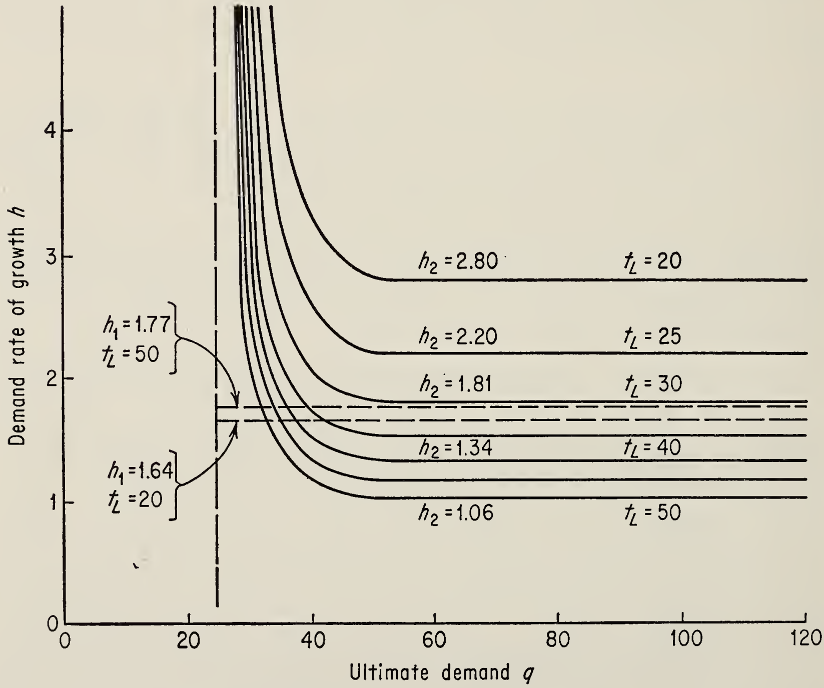

Fig. 5.2. Maximizing probability of success

$q > h_2t_L$. For values of $q < h_2t_L$, equation (5.63) applies and at $q - h_2t_L$ equations (5.62) and (5.63) yield identical results. These curves are shown as solid lines in Fig. 5.2. Because they are so close together, only h_1 values for $t_L = 20$ and for $t_L = 50$ are shown (dashed).

The new firm can succeed only if it can both cover W^* and repay the loan when it is due. Thus the percentage of firms lying above the higher of h_1 or h_2 and to the right of the solid line which rises on the left represents the probability of success for a given value of t_L.

Note that, for $t_L = 20$, h_2 controls the probability and that, as t_L increases, h_2 decreases while h_1 increases. At $t_L \approx 33$ both are equal. Hence this would be the minimal time for which the loan would be desired. For $t_L > 33$, the relevant probability space is controlled by h_1 below and the solid line to the left. As t_L increases beyond 33, probability density is added to the left and subtracted below. The net change for a given probability density function would have to be examined to determine the optimal value of t_L in a specific case.

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